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ANOMALOUS PSEUDOSCALAR-PHOTON VERTEX IN AND OUT OF EQUILIBRIUM

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Abstract

The anomalous pseudoscalar-photon vertex is studied in real time in and out of equilibrium in a constituent quark model. The goal is to understand the in-medium modifications of this vertex, exploring the possibility of enhanced isospin breaking by electromagnetic effects as well as the formation of neutral pion condensates in a rapid chiral phase transition in peripheral, ultrarelativistic heavy-ion collisions. In *equilibrium* the effective vertex is afflicted by infrared and pinch singularities that require hard thermal loop (HTL) and width corrections of the quark propagator. The resummed effective equilibrium vertex vanishes near the chiral transition in the chiral limit. In a strongly *out of equilibrium* chiral phase transition we find that the chiral condensate drastically modifies the quark propagators and the effective vertex. The ensuing dynamics for the neutral pion results in a potential enhancement of isospin breaking and the formation of π^0 condensates. While the anomaly equation and the axial Ward identity are not modified by the medium in or out of equilibrium, the effective *real-time* pseudoscalar-photon vertex is sensitive to low energy physics.

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I. INTRODUCTION

The forthcoming ultrarelativistic heavy ion colliders, RHIC at BNL and LHC at CERN will allow us to probe the physics of the quark gluon plasma, the hadronization and the chiral phase transitions, providing for the first time, experimental access to a phase transition in particle physics as predicted by QCD [1]. Current experiments at CERN SPS are seeking manifestations of these phase transitions with the large acceptance detectors NA49 and WA98, while the large acceptance detector capabilities at RHIC will allow an event-by-event analysis of *both* hadronic and electromagnetic signals. Currently WA98 at SPS-CERN studies fluctuations in the neutral to charged pion ratios on an event-by-event basis [2], and previously the MINIMAX collaboration at Fermilab studied similar signals [3]. These experiments are particularly important in the detection of disoriented chiral condensates (DCC) which are coherent regions of chirally misaligned pion condensates, proposed and discussed by many authors [4], originally within the context of CENTAURO events in cosmic ray experiments [5] but more recently as a possible signal for the chiral phase transition if it occurs out of equilibrium [6–9]. With the turning on of the above mentioned experiments, with their capacity to analyze data from ultrarelativistic heavy-ion collisions on an event-by-event basis, a theoretical exploration of various signals of the chiral phase transition is in order.

The axial anomaly [10] has long been a source of extremely interesting physics, ranging from the explanation of the $\pi^0 \to \gamma\gamma$ decay to understanding the mass splitting between the η and the η' pseudoscalar mesons [11]. In this article we focus on the equilibrium and nonequilibrium aspects of the *anomalous* electromagnetic interaction of the *neutral pion* to investigate the possibility of experimentally important signals arising from this interaction and its modifications in a medium. In particular, since we expect that the collision region will contain a large amount of energy density, a finite temperature study as well as a study of the effects of the axial anomaly in out-of-equilibrium situations will prove important for our understanding of forthcoming experimental results.

Detailed studies in equilibrium have found that the anomalous Ward identity is independent of temperature and chemical potential [12] and that in chiral perturbation theory, valid for $T << T_c$ the amplitude for neutral pion decay into two photons is independent of temperature.

However, in a series of recent articles, Pisarski [13] and Pisarski, Trueman and Tytgat [14] have provided a detailed analysis of the chiral anomaly and the anomalous *vertex* at finite temperature using the Euclidean formulation and concluded, that while the amplitude for the neutral pion decay into two photons at zero temperature is directly proportional to the coefficient of the axial anomaly (given by the anomalous Ward identity), at finite temperatures this relationship no longer holds. This result was anticipated in [15] where it was pointed out that at finite temperature there is a new vector (the four-velocity of the heat bath) which leads to a more complicated tensor structure for the triangle diagram that yields the anomalous amplitude and invalidates the tensor analysis leading to the Sutherland-Veltman theorem [16].

Pisarski [13] also pointed out that near the critical temperature for the chiral phase transition, the amplitude for $\pi^0 \to \gamma\gamma$ *vanishes.* This conclusion was based on an analysis of the corresponding 3-point vertex with *spacelike* external photons, using the imaginary time formalism. Gelis [17] studied the anomalous decay of the neutral pion into two photons at finite temperature and pointed out some subtleties of the triangle diagram in the limit of soft external momenta, and reconciled certain discrepancies between the results of [13,14] and those of [18]. A study in the linear sigma model at finite temperature and chemical potential *in equilibrium* was anticipated in [19] with similar results to those found in [18,17].

In this work we apply the *real time* formulation of quantum field theory to the linear sigma model (LSM) coupled to constituent quarks and photons and calculate the effective coupling between the π^0 and photons both at finite temperature as well as out of equilibrium. This allows us to obtain far more information from the system than could be found from calculations of suitably defined decay amplitudes (for e.g. [17]); in particular, we will be able to construct a real time equation of motion for the neutral pion condensate, and make contact with recent proposals of DCC formation [4,6–9] via anomalous electromagnetic interactions [20].

We first focus our attention on the equilibrium case in the region near the chiral phase transition where the pions and constituent quarks are becoming massless. We find that in the limit of soft photon momenta near the phase transition two different types of divergences appear: infrared divergences in the chirally restored phase associated with the masslessness of the quarks in the triangle diagram, and pinch singularities that arise solely from the finite temperature contributions. The infrared singularities near the chiral limit are the source of discrepancy between the work in [13,14] performed in the imaginary time formulation and that of [18] in the real time version. This discrepancy and its resolution has been recently discussed in [17]. The pinch singularities have a very different origin. For photons of very small virtuality, the quarks that are present in the medium and that can contribute to the triangle diagram, can propagate on-shell. If the quarks are treated as bare objects, then these on-shell states in the thermal bath will propagate without damping and a pinch singularity will appear as a consequence of these undamped on-shell intermediate states.

Both singularities require that the internal quark propagators be dressed by self-energy corrections, giving them a chirally symmetric mass as well as a width of thermal origin. In the Yukawa theory, the fermion receives Hard Thermal Loop (HTL) corrections to its self-energy [17,21] when its momentum $<< g\,T$ where g is the Yukawa coupling and T the temperature. As long as the quark mass derived from the tree-level Yukawa interaction, $m_q >> g\,T$, HTL's are unimportant as far as the real part of the fermion self-energy is concerned. However in the critical region in the strict chiral limit, the constituent quark mass m_q vanishes and the contribution of hard thermal loops (HTL) becomes important. The pinch singularities require a further self-energy resummation that incorporates the on-shell width of the thermal excitations; this goes beyond the HTL contribution.

We will argue that both the hard thermal loop correction to the self-energy as well as the width are dominated by scalar and pseudoscalar exchange. Furthermore, we find that the quark width in the medium arises from the *decay* of the scalar into quark-antiquark

pairs as found previously in [22] within a different context and from collisional contributions in the HTL limit [21]. We compute these corrections to the quark propagators explicitly in the equilibrium case, and we find, in agreement with [13,14,17], that the pion-photon vertex vanishes in the strict chiral limit near the critical temperature. $\pi^0 \to \gamma\gamma$ at finite temperature is, however, of limited relevance for heavy-ion collisions since the typical lifetime of the neutral pion is several orders of magnitude larger than that of the fireball (the latter being $\approx$ 50fm/c). The η meson, however, has a much shorter lifetime $\approx$ 1.2 KeV and finite temperature effects may be of importance for the process $\eta \to \gamma\gamma$ which occurs with a branching ratio of 40%. Finite temperature effects could also be of importance in the cooling of hot and dense stars [18], and in certain models of baryogenesis [23].

Our interest in the *nonequilibrium* calculation of the pion-photon anomalous vertex comes in part from a recent proposal in [20]. It is argued there that in the presence of strong electromagnetic fields, such as might be found in peripheral collisions between heavy nuclei [24], the anomalous coupling might enhance DCC formation along the the neutral pion direction. To quantify the effect of strong EM fields on DCC formation, note that the coupling of the neutral pion to the electromagnetic field is given by $\sim g_{\pi\gamma\gamma}\pi_0 F_{\mu\nu}\tilde{F}^{\mu\nu}$. In peripheral collisions, the strong electromagnetic fields give rise to a *semiclassical* contribution to $F_{\mu\nu}\tilde{F}^{\mu\nu} \propto \vec{E}\cdot\vec{B} \propto Z^2e^2b/R^6$ with b the impact parameter [20]. Therefore, for large impact parameters (very peripheral collisions $b >> 2R$) and heavy nuclei the anomalous interaction of the neutral pion with the electromagnetic field could induce large neutral pion condensates. A further interesting scenario was investigated in [9], namely the possibility of the anomalous vertex inducing photon production via parametric amplification during the chiral phase transition. These investigations studied the phenomenological consequences of the anomaly by using the *vacuum* form for the vertex $g_{\pi\gamma\gamma} \sim \alpha/f_\pi$.

But our previous comments clearly suggest that the vertex *is not* protected from corrections by the medium and the strength of the interaction should be drastically modified by the nonequilibrium state, as is the case at finite temperature. Our aim therefore is : i) to study the nature of these modifications and provide a detailed construction of the vertex in *real time*, and ii) to provide a quantitative assessment of the potential phenomenological consequences of these in-medium corrections. In this paper we deal with the first part of the program, i.e. the more formal aspects of setting up a framework for studying the out-of-equilibrium interactions and relegate to a forthcoming article, the phenomenological and numerical implications of these modifications.

In the nonequilibrium case we assume that the initial state is some thermal state where the chiral order parameter has a small amplitude set by the explicit chiral symmetry breaking term. We then assume that the system undergoes a phenomenological quench and the order parameter ($<\sigma>$) then rolls toward the bottom of its potential. In this case the effective quark mass is then *time dependent* which induces a time dependence in the triangle diagram and hence in the effective anomalous electromagnetic coupling. Furthermore as the $<\sigma>$ evolves, long-wavelength pion fluctuations are generated that feed in to the triangle diagram as well. We obtain the quark propagators during the rolling of the chiral condensate in a mean field approximation. The effective pseudoscalar-photon interactions are then

obtained by explicitly integrating out the quarks and computing the triangle diagram in real-time and a local limit is extracted for quasistatic classical electromagnetic fields. Using this systematic formulation we then derive the equation of motion for the neutral pion field after including both (a) self-consistent mean-field effects and (b) effective nonequilibrium anomalous couplings to the quasi-static semiclassical electromagnetic fields which are of relevance in peripheral heavy-ion collisions. We then argue that unstable long-wavelength pion fluctuations can enhance the isospin breaking electromagnetic effects via the nonequilibrium anomalous coupling.

In an Appendix we also show that the anomaly equation and the axial Ward identity are not modified by the medium in or out of equilibrium. Thus our study shows that the Ward identity does not completely determine the vertex in a medium and more importantly, the latter is sensitive to low energy physics and is therefore model-dependent.

In section II we obtain the effective action in real time in equilibrium, analyze the infrared divergences and obtain the HTL corrections to the self-energy. We analyze the different processes that lead to a width for the fermionic quasiparticles and obtain a quantitative estimate for the local limit of the effective pseudoscalar-photon vertex. [By local limit we mean taking the external momenta to zero].

Section III is devoted to the nonequilibrium case. We consider a 'quenched' chiral phase transition in which the expectation value of the chiral order parameter begins with small amplitude and rolls down its potential hill towards the minimum equilibrium state. The nonequilibrium effective action for pions and photons is obtained in the mean-field approximation. Finally we obtain the effective nonequilibrium vertex for the neutral pion in the limit of quasistatic classical electromagnetic fields, thus obtaining the effective equation of motion for the neutral pion including its anomalous coupling.

Section IV summarizes our results. Appendix A deals with obtaining the anomaly equation in real time both in and out of equilibrium, while Appendix B provides the technical details of the solution of the Dirac equation in a time dependent scalar background.

II. $\pi\gamma\gamma$ VERTEX IN EQUILIBRIUM AT FINITE TEMPERATURE

The purpose of this section is to provide a detailed analysis within real-time, finite temperature field theory, of the effective anomalous coupling between pseudoscalars and photons in a simple model that incorporates all of the essential physics. We work within a Gell-Mann Lévy type model containing a single fermion flavour ψ charged under $U(1)_{em}$, with Yukawa couplings to a scalar σ and a pseudoscalar π. As it will become clear below the results of our analysis generalize straightforwardly to the $U(2) \otimes U(2)$ constituent quark model with the appropriate inclusion of the isospin group structure in the vertices. The Lagrangian density for the model under consideration is given by

$$\mathcal{L} = i\bar{\psi}(\partial\!\!\!/ - ieA\!\!\!/)\psi - 2g\;\bar{\psi}(\sigma + i\pi\gamma_5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_\pi. \tag{2.1}$$

$\mathcal{L}_\pi$ determines the dynamics of the π and σ fields and is a ϕ^4 scalar theory given by,

$$\mathcal{L}_\pi = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \lambda(\sigma^2 + \pi^2 - f_a)^2 + c\sigma \tag{2.2}$$

The vacuum constituent quark mass $\approx$ 300 MeV $\approx 2gf_\pi$ with the vacuum value of $f_\pi = 93$ MeV leads to $g \approx 1.5 > e \approx 0.3$. Although the large phenomenological value of the Yukawa coupling would preclude a perturbative expansion, we will *assume* $1 >> g >> e$ so as to be able to provide a *quantitative* assessment of the in-medium effects at least within a perturbative expansion. While the assumption $g << 1$ is necessary if we want to provide a quantitative understanding based on perturbation theory, the condition $e << g$ is imposed only to simplify the calculations and can be easily relaxed without affecting the qualitative validity of the results in what follows.

A small linear term $c\,\sigma$ ensures that in equilibrium isospin symmetry is broken along the σ direction. At zero temperature $\langle\sigma\rangle \approx f_\pi$ and the explicit symmetry breaking term gives the pions their mass. The strict chiral limit refers to $c \equiv 0$. At finite temperatures $T \neq 0$, the expectation value of σ acquires temperature dependence and we split the quantum field into a c-number expectation value and quantum fluctuations:

$$\sigma(x) = \sigma_0(T) + \chi(x)\ ; \quad \langle\sigma(x)\rangle = \sigma_0(T), \tag{2.3}$$

Here the expectation value of the Heisenberg operators is evaluated in the thermal density matrix. The fermion thus gets a temperature dependent tree-level mass,

$$m_q(T) = 2g\ \sigma_0(T). \tag{2.4}$$

Our subsequent analysis will be valid in the regime where $T >> g\ \sigma_0(T)$. In particular this condition will be satisfied near the critical temperature where in the strict chiral limit $c = 0$, $\sigma_0(T)$ vanishes as $\sigma_0(0)\ \sqrt{1-\frac{T}{T_c}}\ ;\ T \to T_c^-$.

A. The real time effective action:

Within the simple constituent quark model introduced above, we now want to study the effective theory of pions interacting with photons at a finite temperature $T \neq 0$. Moreover, we would like to be able to study the *dynamics* of such interactions. *Static* phenomena can be studied using the imaginary time formulation of finite temperature field theory.

The proper way to study *dynamical* phenomena in quantum field theory is via the **real time** evolution of an initially prepared density matrix. This is implemented in the path integral formulation by use of the Closed Time Path (CTP) formalism of Keldysh and Schwinger [25,26]. This formalism allows us to compute the time evolution of expectation values of Heisenberg operators and can be applied both to real time equilibrium calculations as well as nonequilibrium ones. In this formulation the action is defined on a contour that runs forward and backward in time as befits the time evolution of an initially prepared density matrix. Field operators on the forward and backward branches are denoted with $+$ and $-$ superscripts respectively and are independent field variables. For a description of this formulation within the setting of problems akin to those studied here see [25–29].

As we have stressed in the introduction our goal is to derive an effective description for the pions and photons which really means obtaining the real time effective action by integrating out the quarks including the anomalous pion-photon interactions. We remark that, for our purposes the *decay amplitude* for $\pi \to \gamma\gamma$ as defined in [17] for example, is a more restricted quantity than the full real-time effective action. The former is given by a particular cut of the self-energy graph (Fig. 2) for the π. The couplings in the effective action also contain discontinuities arising from the on-shell propagation of intermediate fermions in the thermal bath. Cross-couplings between fields living on the forward and backward contours will also be induced. *All* of these terms in the effective action will be important for understanding the effective pion-photon interactions in the thermal bath; for example the full real-time equation of motion for pion condensates will necessarily receive contributions from all these different terms.

Although the effective action in Euclidean time has been derived previously, we obtain here the real time effective action, which we believe is a novel approach. The utility of the real time effective action lies in the fact that it allows us to obtain the real-time, retarded equations of motion for expectation values even *out of equilibrium.* As emphasized above the focus of this article is to provide a real-time description of non-equilibrium phenomena. Since the real time effective action within this setting has not yet received attention, we begin by studying the equilibrium situation first to compare with previous results in the literature and then continue with the focus of our program, the non-equilibrium aspects.

In order to obtain the aforementioned effective couplings we integrate out the fermions to one-loop to get the (CTP) effective action for the pseudoscalars and the photons up to this order. Formally this is a rather straightforward procedure, the full real-time effective action being given by:

$$
\begin{aligned}
&\exp iS_{eff}[A^{\pm}_{\mu},\pi^{\pm}] = \\
&= \int [\mathcal{D}\psi^{\pm}][\mathcal{D}\bar{\psi}^{\pm}] \exp[i \int d^3x\; dt\{i\bar{\psi}^{+}(i\partial\!\!\!/ - 2g\sigma_0)\psi^{+} - (+\longrightarrow -)\}] \times \qquad (2.5)\\
&\times \exp[i \int d^4x\{e\bar{\psi}^{+}\gamma^{\mu}A^{+}_{\mu}\psi^{+} - 2g\bar{\psi}^{+}(\chi^{+} + i\pi\gamma_5)\psi^{+} - (+\longrightarrow -)\}].
\end{aligned}
$$

The one-loop contribution is obtained by simply expanding the interaction part $\exp[iS_{int}]$ to $O(e^2 g)$ and integrating out the fermions yielding the following expression for the $\pi\gamma\gamma$ interactions:

$$
\begin{aligned}
&S_{\pi\gamma\gamma} = -2ge^2 \int d^4x\; d^4x_1\; d^4x_2 \times \\
&\times \Big\{\pi^{+}(x)A^{+}_{\mu}(x_1)A^{+}_{\nu}(x_2)\;\; \mathrm{Tr}[S^{++}(x_2,x)\gamma_5 S^{++}(x,x_1)\gamma^{\mu}S^{++}(x_1,x_2)\gamma^{\nu}] \qquad (2.6)\\
&-\pi^{+}(x)A^{+}_{\mu}(x_1)A^{-}_{\nu}(x_2)\;\; \mathrm{Tr}[S^{-+}(x_2,x)\gamma_5 S^{++}(x,x_1)\gamma^{\mu}S^{+-}(x_1,x_2)\gamma^{\nu}]
\end{aligned}
$$

$$-\pi^{+}(x)A^{-}_{\mu}(x_1)A^{+}_{\nu}(x_2)\ \ \mathrm{Tr}[S^{++}(x_2,x)\gamma_5 S^{+-}(x,x_1)\gamma^{\mu}S^{-+}(x_1,x_2)\gamma^{\nu}]$$

$$+\pi^{+}(x)A^{-}_{\mu}(x_1)A^{-}_{\nu}(x_2)\ \ \mathrm{Tr}[S^{-+}(x_2,x)\gamma_5 S^{+-}(x,x_2)\gamma^{\nu}S^{--}(x_2,x_1)\gamma^{\mu}] - (\pi^{+}\to\pi^{-})\Big\}\ .$$

where S^{-+}, S^{+-}, S^{++} and S^{--} are the real time fermionic propagators given by:

$$S^{\pm\pm}(x,x') = -i\,\langle\psi^{\pm}(x)\bar{\psi}^{\pm}(x')\rangle; \tag{2.7}$$

$$S^{-+}(x,x') = -i\,\langle\psi^{-}(x)\bar{\psi}^{+}(x')\rangle; \qquad S^{+-}(x,x') = i\,\langle\bar{\psi}^{+}(x')\psi^{-}(x)\rangle; \tag{2.8}$$

$$S^{++(--)}(x,x') = -S^{-+(+-)}(x,x')\,\Theta(x_0-x_0') - S^{+-(-+)}(x-x')\,\Theta(x_0'-x_0). \tag{2.9}$$

Note that the interactions have induced cross-couplings between fields on different branches of the CTP contour. This means that answers to physical questions such as how the pion couples to classical electromagnetic sources will be affected by the interplay between all these cross-couplings in the effective action. In particular the equation of motion for the expectation value of the pion field will require *all* of these contributions [27,28]. Evaluation of the various terms in the effective action is more convenient in momentum space and in the equilibrium case we can introduce the Fourier-transformed fields and propagators:

$$\tilde{\pi}(k) = \int d^4x\ e^{ik\cdot x}\ \pi(x)\ , \tag{2.10}$$

$$\tilde{A}_{\mu}(k) = \int d^4x\ e^{ik\cdot x}\ A_{\mu}(x)\ , \tag{2.11}$$

$$S(k) = \int d^4x\ e^{ik\cdot(x-x')}\ S(x,x')\ . \tag{2.12}$$

The effective action in momentum space then takes the following form

$$S_{\pi\gamma\gamma} = -2e^2 g\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\times \tag{2.13}$$

$$\times\Big\{\tilde{\pi}^{+}(-p_1-p_2)\tilde{A}^{+}_{\mu}(p_1)\tilde{A}^{+}_{\nu}(p_2)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S^{++}(k-p_1)\gamma_5 S^{++}(k+p_2)\gamma^{\mu}S^{++}(k)\gamma^{\nu}]+$$

$$-\tilde{\pi}^{+}(-p_1-p_2)\tilde{A}^{+}_{\mu}(p_1)\tilde{A}^{-}_{\nu}(p_2)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S^{-+}(k-p_1)\gamma_5 S^{++}(k+p_2)\gamma^{\mu}S^{+-}(k)\gamma^{\nu}]+$$

$$-\tilde{\pi}^{+}(-p_1-p_2)\tilde{A}^{-}_{\mu}(p_1)\tilde{A}^{+}_{\nu}(p_2)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S^{++}(k-p_1)\gamma_5 S^{+-}(k+p_2)\gamma^{\mu}S^{-+}(k)\gamma^{\nu}]+$$

$$+\tilde{\pi}^{+}(-p_1-p_2)\tilde{A}^{-}_{\mu}(p_1)\tilde{A}^{-}_{\nu}(p_2)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S^{-+}(k-p_1)\gamma_5 S^{+-}(k+p_2)\gamma^{\mu}S^{--}(k)\gamma^{\nu}]\Big\}.$$

¿From the definitions of the real time propagators (2.9) it is clear that the four propagators satisfy the identity:

$$S^{++} + S^{+-} + S^{-+} + S^{--} = 0, \tag{2.14}$$

which of course continues to hold for the Fourier-transformed propagators as well. So we can rewrite them in terms of three independent functions in the Keldysh [26] notation,

$$S_R(k) = S^{++}(k) + S^{+-}(k), \tag{2.15}$$

$$S_A(k) = S^{++}(k) + S^{-+}(k), \tag{2.16}$$

$$S_F(k) = S^{++}(k) + S^{--}(k). \tag{2.17}$$

Here S_R, S_A are the retarded and advanced propagators, while S_F is a symmetric combination which is closely related to the spectral density in thermal equilibrium:

$$S_F(k) = [1 - 2n_f(|k_0|)]\mathrm{sgn}(k_0)[S_R(k) - S_A(k)] \tag{2.18}$$

$$n_f(|k_0|) = \frac{1}{e^{\beta|k_0|} + 1}. \tag{2.19}$$

This relationship can be shown to arise via the Lehmann spectral representation for states in thermal equilibrium and it does not hold out-of-equilibrium. All these definitions and relations are exact and continue to hold for the full field theory in thermal equilibrium.

B. The 'bare' $\pi\gamma\gamma$ vertex:

The anomalous couplings in the CTP effective action are to be thought of in terms of the triangle diagram in Fig. 1 with insertions of external fields at the vertices, carrying all possible combinations of + and − labels on them, corresponding to the relevant portions of the CTP contour [27–29].

Since we want to study the low-energy effective couplings, we must first clearly identify the scales involved in the problem. The external photons have 4-momenta p_1, p_2 (see Fig. 1 for example) while the pion has momentum $-p_1 - p_2$ by momentum conservation. Henceforth we will collectively denote the momenta of the external fields by P_{ext}. By low-energy vertices we will always mean effective interactions where $P_{\text{ext}} << T$, and since we want to remain close to the chiral phase transition, we choose $m_q << T$ as well. From our subsequent analysis it will become clear that other scales such as gT, g^2T, originating dynamically from the thermal contributions to the fermion self-energy will also be extremely important. (Recall, in this connection our initial assumption that $g << 1$ so that there is indeed a possible hierarchy of scales T, gT.., etc.)

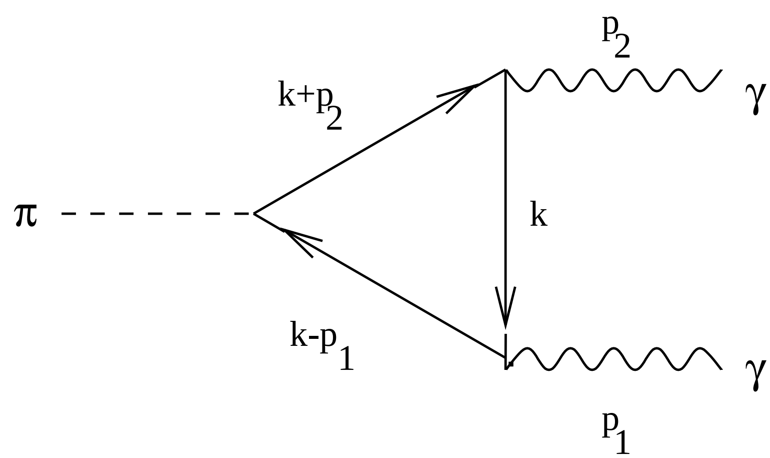

FIG. 1. The triangle diagram at finite temperature with bare fermion propagators in the internal lines.

First consider the regime $gT << m_q << T$ and $P_{\rm ext} << m_q$. In order to provide motivation for the other scales that will be important at finite T let us first calculate the 'bare' $\pi\gamma\gamma$ vertex, where by 'bare' we are referring to an evaluation of the terms in the effective action in Eq. (2.6) with *undressed* finite temperature propagators for the fermions in the loop (Fig. 2). At finite temperature the free fermion Green's functions in the real-time approach are:

$$S^{\pm\pm}(k) = (\not k + m_q)\left[\pm\frac{1}{k^2 - m_q^2 \pm i\epsilon} + 2\pi i\; n_f(\omega_k)\delta(k^2 - m_q^2)\right] \tag{2.20}$$

$$S^{\mp\pm}(k) = 2\pi i\;(\not k + m_q)\delta(k^2 - m_q^2)\left[\Theta(\pm k_0) - n_f(\omega_k)\right] \tag{2.21}$$

$$n_f(|k_0|) = \frac{1}{e^{\beta k_0} - 1}; \quad \omega_k = \sqrt{k^2 + m_q^2}. \tag{2.22}$$

The local form of the low energy vertex $\pi^+ A^+ A^+$ (from Eq. (2.13)) is obtained by setting $P_{\rm ext} = 0$ in the computation of the triangle diagram, (as was done in the imaginary time formulation in [13,14,31]). However, in the real time formulation, setting the external momenta to zero inside the quark loop leads to a *divergent* result:

$$S^{+++}_{\pi\gamma\gamma} = 8m_q\, e^2 g \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} \times \Big\{ i\,\epsilon^{\alpha\beta\mu\nu}\; p_{1\alpha}p_{2\beta}\;\tilde{\pi}^+(-p_1 - p_2)\tilde{A}^+_\mu(p_1)\tilde{A}^+_\nu(p_2) \tag{2.23}$$

$$\int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2 - m_q^2 + i\epsilon} + 2\pi i\; n_f(\omega_k)\;\delta(k^2 - m_q^2)\right]^3 \Big\}$$

This integral has two different sources of infrared divergences. In the chiral limit $m_q \to 0$ the temperature independent term becomes sensitive to the infrared modes. At finite

temperature this divergence must be cured by resumming the contributions from HTL. On the other hand the temperature dependent terms lead to divergences $\sim T^2/\epsilon^2$. This is a pinch singularity that originates from the propagation of on-shell fermionic intermediate states present in the bath. At tree level, these quarks in the plasma propagate without damping over arbitrarily long times. However, in an interacting plasma the quarks will acquire a width Γ_k and will only propagate during time scales $\approx 1/\Gamma_k$. Including this effect in the fermion propagator will cut off these pinch singularities.

This naive calculation is obviously flawed but clearly indicates that using bare propagators and setting P_{ext} to zero is inconsistent with the high temperature limit. If we imagine taking P_{ext} to zero continuously, at some value the correlator will become sensitive to thermal corrections to the self-energy. The real part of the thermal self-energy should play the role of m_q in the loop while the on-shell thermal width should replace the Feynman parameter ϵ.[1]

Before proceeding to a more consistent description using resummed propagators, we seek to establish contact with the results of [13] and [17] using the real time effective action obtained above. We do the calculation by keeping the external momenta non-zero, and then take the zero momentum limit smoothly at the end of the calculation.

Using bare fermion propagators and defining

$$\omega = \sqrt{k^2 + m_q^2}\ ; \ \ \omega_1 = \sqrt{(\vec{k} - \vec{p}_1)^2 + m_q^2}\ ; \ \ \omega_2 = \sqrt{(\vec{k} + \vec{p}_2)^2 + m_q^2}\ ; \tag{2.24}$$

$$n = n_f(\omega)\ ; \ \ n_1 = n_f(\omega_1)\ ; \ \ n_2 = n_f(\omega_2)\ , \tag{2.25}$$

we find that the $\pi^+ A^+_\mu A^+_\nu$ coupling in the effective action is given by:

$$S^{+++}_{\pi\gamma\gamma} = -m_q\, e^2 g \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} \times \epsilon^{\alpha\beta\mu\nu}\ p_{1\alpha}\ p_{2\beta}\ \tilde{\pi}^+(-p_1-p_2)\ \tilde{A}^+_\mu(p_1)\ \tilde{A}^+_\nu(p_2) \times \tag{2.26}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega\omega_1\omega_2}\Big\{(1-n_1)(1-n_2)n\Big[\frac{1}{(\omega_1-\omega-p_1^0-i\epsilon)(\omega_2-\omega+p_2^0-i\epsilon)}+$$

$$\frac{1}{(\omega_2-\omega-p_2^0-i\epsilon)(\omega_1-\omega+p_1^0-i\epsilon)}+\frac{1}{(\omega_2-\omega-p_2^0-i\epsilon)(\omega_2+\omega_1-p_2^0-p_1^0-i\epsilon)}+$$

$$\frac{1}{(\omega_1-\omega-p_1^0-i\epsilon)(\omega_1+\omega_2-p_1^0-p_2^0-i\epsilon)}+\frac{1}{(\omega_1-\omega+p_1^0-i\epsilon)(\omega_1+\omega_2+p_1^0+p_2^0-i\epsilon)}+$$

$$\frac{1}{(\omega_2-\omega+p_2^0-i\epsilon)(\omega_1+\omega_2+p_1^0+p_2^0-i\epsilon)}\Big] - n_1n_2(1-n)\,(\text{c.c.})+$$

[1]One of us (S.P.K.) would like to thank S. Jeon and L. Yaffe for pointing this out.

$$n_1(1-n_2)(1-n)\left[\frac{1}{(\omega-\omega_1+p_1^0-i\epsilon)(\omega+\omega_2-p_2^0-i\epsilon)}+\right.$$
$$\frac{1}{(\omega+\omega_2+p_2^0-i\epsilon)(\omega-\omega_1-p_1^0-i\epsilon)}+\frac{1}{(\omega_2+\omega+p_2^0-i\epsilon)(\omega_2-\omega_1+p_1^0+p_2^0-i\epsilon)}+$$
$$\frac{1}{(\omega-\omega_1+p_1^0-i\epsilon)(\omega_2-\omega_1+p_1^0+p_2^0-i\epsilon)}+\frac{1}{(\omega-\omega_1-p_1^0-i\epsilon)(\omega_2-\omega_1-p_1^0-p_2^0-i\epsilon)}+$$
$$\left.\frac{1}{(\omega_2+\omega-p_2^0-i\epsilon)(\omega_2-\omega_1-p_1^0-p_2^0-i\epsilon)}\right]-(1-n_1)n_2n\,[\text{c.c.}]+$$
$$(1-n_1)n_2(1-n)\left[\frac{1}{(\omega_1+\omega-p_1^0-i\epsilon)(\omega-\omega_2+p_2^0-i\epsilon)}+\right.$$
$$\frac{1}{(\omega-\omega_2-p_2^0-i\epsilon)(\omega+\omega_1+p_1^0-i\epsilon)}+\frac{1}{(\omega-\omega_2-p_2^0-i\epsilon)(\omega_1-\omega_2-p_1^0-p_2^0-i\epsilon)}+$$
$$\frac{1}{(\omega+\omega_1-p_1^0-i\epsilon)(\omega_1-\omega_2-p_1^0-p_2^0-i\epsilon)}+\frac{1}{(\omega+\omega_1+p_1^0-i\epsilon)(\omega_1-\omega_2+p_1^0+p_2^0-i\epsilon)}+$$
$$\left.\frac{1}{(\omega-\omega_2+p_2^0-i\epsilon)(\omega_1-\omega_2+p_1^0+p_2^0-i\epsilon)}\right]-n_1(1-n_2)n\,[\text{c.c.}]+$$
$$n_1n_2n\left[\frac{1}{(\omega+\omega_1+p_1^0+i\epsilon)(\omega+\omega_2-p_2^0+i\epsilon)}+\right.$$
$$\frac{1}{(\omega+\omega_2+p_2^0+i\epsilon)(\omega+\omega_1-p_1^0+i\epsilon)}+\frac{1}{(\omega+\omega_2+p_2^0+i\epsilon)(\omega_1+\omega_2+p_1^0+p_2^0+i\epsilon)}+$$
$$\frac{1}{(\omega+\omega_1+p_1^0+i\epsilon)(\omega_1+\omega_2+p_1^0+p_2^0+i\epsilon)}+\frac{1}{(\omega+\omega_1-p_1^0+i\epsilon)(\omega_1+\omega_2-p_1^0-p_2^0+i\epsilon)}+$$
$$\left.\frac{1}{(\omega+\omega_2-p_2^0+i\epsilon)(\omega_1+\omega_2-p_1^0-p_2^0+i\epsilon)}\right]-(1-n_1)(1-n_2)(1-n)\,[\text{c.c.}]\Big\}$$

The above interaction is just one of the couplings that appears in the real time effective action (others are $\pi^+A^+_\mu A^-_\nu, \pi^+A^-_\mu A^-_\nu$, etc.). An important point to note is that aside from the direct interactions with photons it also contains discontinuities from the on-shell propagation of fermionic intermediate states that appear in the bath and couple to the external electromagnetic field. These are encoded in the $i\epsilon$ terms of the energy denominators and give rise to pinch singularities when external momenta are set to zero as previously discussed.

We now make contact with the results of [13,17,18] by calculating the decay amplitude $\Gamma(\pi\to\gamma\gamma)$ which can be obtained from the above expression by simply *ignoring* the discontinuities in this effective vertex. The resulting expression can be interpreted as a particular

cut of the 3-loop self-energy graph [17] (see Fig. 2) for the π where the photons appear as final states.

In the limit of $P_{\text{ext}} << T$ and $P_{\text{ext}} << m_q$, the distribution functions and the energy-denominators can be expanded in powers of P_{ext}/T and P_{ext}/m_q respectively to yield the amplitude (or the imaginary part of the retarded self-energy) for $\pi \to \gamma\gamma$. After long-winded algebraic manipulations we obtain the following form for the vertex in the zero-momentum limit:

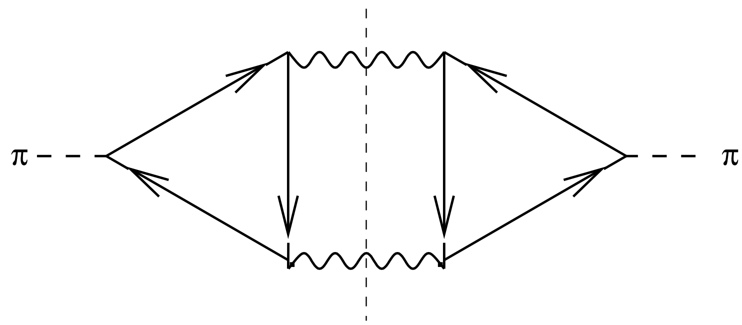

FIG. 2: The 3-loop self-energy cut to yield the required decay amplitude in [17].

$$\Gamma(\pi \to \gamma\gamma) = m_q\, e^2 g \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega^3} \times \tag{2.27}$$

$$\left\{ \frac{3(1-2n)}{2\omega^2} + \frac{dn(\omega)}{d\omega} \frac{\vec{k}\cdot\vec{p}_1}{2\omega} \left[\frac{1}{\tilde{k}\cdot p_1} - \frac{1}{k\cdot p_1} + \frac{1}{\tilde{k}\cdot(p_1+p_2)} - \frac{1}{k\cdot(p_1+p_2)} + \right.\right.$$

$$\left. + \frac{2\vec{k}\cdot\vec{p}_1}{k\cdot p_1 k\cdot(p_1+p_2)} + \frac{2\vec{k}\cdot\vec{p}_1}{\tilde{k}\cdot p_1\tilde{k}\cdot(p_1+p_2)} - \frac{2\vec{k}\cdot\vec{p}_2}{k\cdot p_2 k\cdot(p_1+p_2)} - \frac{2\vec{k}\cdot\vec{p}_2}{\tilde{k}\cdot p_2\tilde{k}\cdot(p_1+p_2)} \right]$$

$$+ \frac{dn(\omega)}{d\omega} \frac{\omega^2\vec{p}_1^{\,2} - (\vec{k}\cdot\vec{p}_1)^2}{2\omega} \left[\frac{\vec{k}\cdot\vec{p}_1}{(\tilde{k}\cdot p_1)^2\tilde{k}\cdot(p_1+p_2)} - \frac{\vec{k}\cdot\vec{p}_1}{(k\cdot p_1)^2 k\cdot(p_1+p_2)} \right.$$

$$+ \frac{\vec{k}\cdot\vec{p}_1}{\tilde{k}\cdot p_1(\tilde{k}\cdot(p_1+p_2))^2} - \frac{\vec{k}\cdot\vec{p}_1}{k\cdot p_1(k\cdot(p_1+p_2))^2} - \frac{\vec{k}\cdot\vec{p}_2}{\tilde{k}\cdot p_2(\tilde{k}\cdot(p_1+p_2))^2}$$

$$\left. + \frac{\vec{k}\cdot\vec{p}_2}{k\cdot p_2(k\cdot(p_1+p_2))^2} - \frac{1}{k\cdot p_1 k\cdot(p_1+p_2)} - \frac{1}{\tilde{k}\cdot p_1\tilde{k}\cdot(p_1+p_2)} \right]$$

$$\left. - \frac{1}{2}\frac{d^2n(\omega)}{d\omega^2}(\vec{k}\cdot\vec{p}_1)^2 \left[\frac{1}{k\cdot p_1 k\cdot(p_1+p_2)} + \frac{1}{\tilde{k}\cdot p_1\tilde{k}\cdot(p_1+p_2)} \right] + (p_1 \leftrightarrow p_2) \right\}$$

$$\text{where } \omega = \sqrt{\vec{k}^2 + m_q^2} \ , \quad \tilde{k} = (\omega, -\vec{k}) \ . \tag{2.28}$$

The symmetrization with respect to p_1 and p_2 applies only to the momentum-dependent terms. This is one of the important results of this paper. Using this expression one can calculate the required amplitude for different external kinematical configurations. For spacelike photons $(p_1)^0 = (p_2)^0 = 0$ and in the limit $m_q/T \to 0$ the expression turns out to be finite and is given by:

$$\Gamma(\pi \to \gamma\gamma) = m_q \, e^2 g \int \frac{d^3k}{(2\pi)^3} \, \frac{1}{k^3} \left\{ \frac{3}{2k^2}[1 - 2n(k)] + \frac{3}{k} \, \frac{dn(k)}{dk} - \frac{d^2n(k)}{dk^2} \right\} \ . \tag{2.29}$$

This expression agrees with the result of [17] and after careful integrations by parts can be easily shown to yield the zeta-function dependent result of the Euclidean calculation. It behaves as $\sim m_q \, e^2 \, g/T^2$ in agreement with [13,14].

In the case of pion decay at rest to two back-to-back on-shell photons we find after straightforward substitution

$$\begin{aligned} &\Gamma(\pi \to \gamma\gamma) = \\ &m_q \, e^2 g \int \frac{d^3k}{(2\pi)^3} \, \frac{1}{\omega^3} \left\{ \frac{3}{2\omega^2}[1 - 2n(\omega)] - \frac{1}{\omega} \, \frac{dn(\omega)}{d\omega} - \frac{d^2n(\omega)}{d\omega^2} \frac{(\vec{k}\cdot\vec{p})^2}{\omega^2 p^2 - (\vec{k}\cdot p)^2} \right\} \ . \end{aligned} \tag{2.30}$$

This coincides with the expression found in [17] and can be shown to be equivalent to the results obtained in [18]. In this case and also for other generic kinematical configurations Eq.(2.27) in fact behaves as $\sim e^2 \, g/T$. These results are, however, valid only in the regime where $T >> m_q >> g\,T$ so that we can ignore the hard thermal loops.

While this calculation seems to indicate that self-energy corrections will not play a role in the amplitude, the terms in the effective action also involve additional contributions from thermal discontinuities. These are the terms that yield pinch singularities in the zero momentum limit, as can be easily checked by setting the external momenta *equal* to zero in the exact expression for $S^{+++}_{\pi\gamma\gamma}$ Eq. (2.26).

The result obtained for the finite temperature vertex in the imaginary time formulation in [13,14,31] cannot be extrapolated to the real time domain. In [13,14] the calculation was performed by setting the external Matsubara frequencies to zero. In order to obtain the real time behavior, the external Matsubara frequencies must be kept non-zero in the evaluation of the triangle diagram and the resulting expression will therefore be a function of the external frequencies ω_n defined at the discrete points along the imaginary axis in the frequency plane. The real-time vertex is then obtained by the continuation $\omega_n \to i(\omega \pm i\epsilon)$; the imaginary parts resulting from the $\pm i\epsilon$ will reveal the different discontinuities and therefore the ensuing pinch singularities in the limit of soft external four-momentum.

C. The dressed vertex with HTL resummation:

From the above expressions it is clear that the amplitude is sensitive to momenta $\sim m_q$ and as m_q/T is taken to zero we have to include self-energy corrections to the quark propagator. Furthermore we have seen that the interaction terms in the effective action also include the discontinuities from the quarks in the loop and these in turn will be sensitive to the thermal width on-shell when the external momenta are sufficiently small (i.e. $P_{\rm ext} << \Gamma$ where Γ is the on-shell width for the fermionic quasiparticles.).

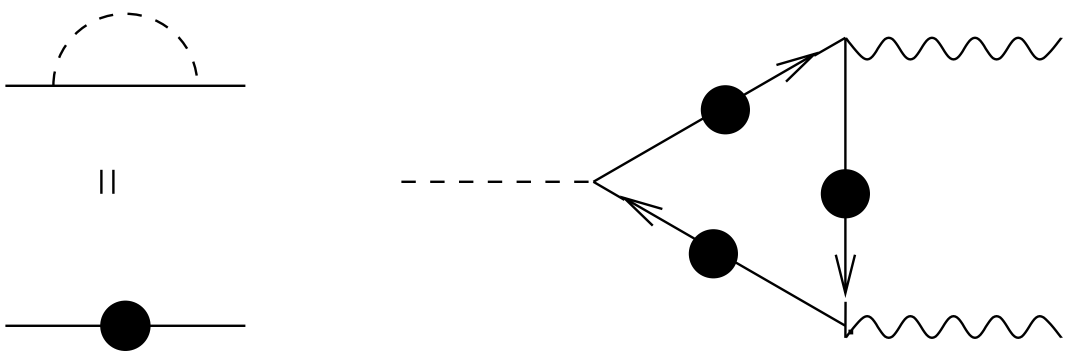

FIG. 3 . The triangle diagram at finite temperature with resummed fermion propagators in the loops.

In the high temperature, long-wavelength limit, i.e. when m_q ; $P_{\rm ext} << gT$ HTL resummed propagators must be introduced for the quarks in the loop (see Fig. 3). The theory obtains HTL corrections both from the gauge bosons and the scalar sector invoving the σ and the π. However, since we have chosen $e << g$ we will ignore the contributions from the photons to HTL. The σ and the π contribute equally to the thermal self-energy for the soft quarks. In addition there are also the usual zero temperature logarithmic corrections to the fermion mass m_q which we shall ignore since these are not relevant for the discussion. The retarded self-energy corrections from the scalars were obtained in [22] and we summarize the main ingredients that are relevant to our discussion here, in the limit $m_q = 0$. The one loop retarded self-energy for the quarks is given by $\Sigma(k_0 - i\epsilon, \vec{k}) = \Sigma^\sigma(k_0 - i\epsilon, \vec{k}) + \Sigma^\pi(k_0 - i\epsilon, \vec{k})$ with

$$\Sigma^a(k_0 - i\epsilon, \vec{k}) = i\,\gamma_0\, k_0\, \tilde{\varepsilon}^{(0)}_{\vec{k}}(k_0 - i\epsilon) + \vec{\gamma}\cdot\vec{k}\;\; \tilde{\varepsilon}^{(1)}_{\vec{k}}(k_0 - i\epsilon). \tag{2.31}$$

for $a = \sigma, \pi$. We define

$$\left\{ \begin{array}{c} \tilde{\varepsilon}^{(0)}_{\vec{k}}(k_0 - i\epsilon) \\ \tilde{\varepsilon}^{(1)}_{\vec{k}}(k_0 - i\epsilon) \end{array} \right\} = \int ds \frac{1}{(k_0 - i\epsilon)^2 + s^2} \left\{ \begin{array}{c} \rho^{(0)}_{\vec{k}}(s) \\ s\,\rho^{(1)}_{\vec{k}}(s) \end{array} \right\} \tag{2.32}$$

using the one-loop spectral densities given by the expressions:

$$\rho^{(0)}_{\vec{k}}(s) = 4g^2 \int \frac{d^3q}{(2\pi)^3} \frac{\bar{\omega}_q}{2\omega_{k+q}\bar{\omega}_q} \times$$
$$[\delta(s-\omega_{k+q}-\bar{\omega}_q)(1+n_{k+q}-\bar{n}_q)+\delta(s-\omega_{k+q}+\bar{\omega}_q)(n_{k+q}+\bar{n}_q)] \ ,$$

$$\rho^{(1)}_{\vec{k}}(s) = 4g^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{k+q}\bar{\omega}_q} \frac{\vec{k}\cdot\vec{q}}{k^2} \times$$
$$[\delta(s-\omega_{k+q}-\bar{\omega}_q)(1+n_{k+q}-\bar{n}_q)-\delta(s-\omega_{k+q}+\bar{\omega}_q)(n_{k+q}+\bar{n}_q)] \, , \tag{2.33}$$

where $n_{\vec{k}+\vec{q}}$, $\bar{n}_q$ are the Bose-Einstein and Fermi-Dirac distribution functions and ω , $\bar{\omega}$ are the bosonic and fermionic frequencies respectively. The Bose-Einstein distributions for the σ and π fields are functions of their respective masses. We have also neglected the ultraviolet counterterms as these are irrelevant for our discussion.

The terms proportional to $\delta(s-\omega_{k+q}-\bar{\omega}_q)$ in the spectral densities correspond to the processes $q \leftrightarrow \phi + q$, where ϕ is either a σ or π field, and determine the usual two particle cut that survives in the zero temperature limit. The terms proportional to $\delta(s-\omega_{k+q}+\bar{\omega}_q)$ correspond to Landau damping and processes of the type $\phi \leftrightarrow q + \bar{q}$, which can only occur in the medium. Near the critical temperature and in the strict chiral limit, $m_\sigma, m_\pi \to 0$ as $T \to T_c^-$ and $m_q \to 0$. Therefore, in this limit, the Landau damping terms lead to a HTL contribution to the quark propagator.

We extract the HTL contribution to the quark self-energy in the strict chiral limit and near the critical temperature so that $T >> m_q, m_\sigma, m_\pi, P_{\mathrm{ext}}$ in the usual manner [32–36] and find

$$\Sigma_{HTL}(k_0 - i\epsilon; \vec{k}) = -\frac{M^2_{eff}}{k} \left\{ \gamma_0 \ln \frac{k_0+k-i\epsilon}{k_0-k-i\epsilon} + \vec{\gamma}\cdot\hat{k}\left[2 - \frac{k_0}{k}\ln\frac{k_0+k-i\epsilon}{k_0-k-i\epsilon}\right]\right\}$$
$$M^2_{eff} = \frac{g^2\,T^2}{8} \tag{2.34}$$

This HTL contribution is very similar to that of the gauge fields. In fact including the gauge field is tantamount to replacing $g^2 \to g^2 + e^2$ in the above expression.

In the HTL approximation, the effective quark propagator has poles describing two branches of collective modes [32,33]. In this approximation the self energy only has an absorptive part below the light cone, so that to leading order in the HTL limit the on-shell width of the collective modes vanishes. The effective mass for the collective modes M^2_{eff} does not break chiral symmetry [32–36] but serves as an infrared regulator. In order to obtain the width of the quasiparticles to cure the pinch singularities we must go beyond the HTL contributions.

Although including the damping of excitations in the medium and consequently a width in their spectral densities is a physically reasonable manner to cure the pinch singularities, an alternative approach has been discussed in the literature [37].

D. Including the width - beyond HTL:

There are several different contributions to the thermal width of the quark in the medium. These come from scalars, pseudoscalars and gauge bosons and have very different physical origins. We study separately the contributions from the scalar and pseudoscalars and of gauge bosons to clarify the differences.

1. Scalar/pseudoscalar contributions:

In the expression for the one loop self-energy given above (2.31)-(2.33) the terms proportional to $\delta(s-\omega_{k+q}+\bar{\omega}_q)$ *could* lead to an imaginary part on the mass shell of the collective modes. This would be the case if the delta function is satisfied for $s=\omega_{\pm}(k)$ with $\omega_{\pm}(k)$ being the dispersion relation for the different branches of collective modes. This relation clearly depends on the mass of the scalar or pseudoscalar particle in the loop. Since this delta function arises from the process $\phi \leftrightarrow q+\bar{q}$ the constraint is satisfied if the scalar (or pseudoscalar) boson **can decay** into a $q\bar{q}$ pair.

The fact that the *decay* of the scalar leads to a thermal width for the fermion has been previously recognized in [22] and can be understood from a simple kinetic argument: the change in time of the spin-averaged quark distribution can be written as a balance equation with a gain term minus a loss term. The gain term arises at lowest order from the decay of the scalar into $q\bar{q}$ pairs with a factor

$$|\mathcal{M}_{fi}|^2 n_{k+q}(1-\bar{n}_k)(1-\bar{n}_q),$$

while the loss term arises from the 'recombination' of quark-antiquark pairs into a sigma meson with probability

$$|\mathcal{M}_{fi}|^2(1+n_{k+q})\bar{n}_k\bar{n}_q$$

with $|\mathcal{M}_{fi}|^2$ the usual transition matrix element. The linearization of the resulting Boltzmann equation leads to twice the damping rate [22] of the quark of momentum k. Thus the decay of the scalar into $q\bar{q}$ pairs in the medium induces a width for the quark or the collective excitations [22].

In the strict chiral limit with $T << T_c$, the mass of the σ meson ≈ 660 MeV while that of the constituent quark is ≈ 300 MeV so that the kinematical decay condition *could* be fulfilled. Near the critical temperature the sigma mass vanishes as does the constituent quark mass (in the strict chiral limit) but now for $T >> m_q$ the important mass scale is determined by M_{eff} which for $g \approx 1.5$, $T \approx T_c \approx 150$ MeV yields $M_{eff} \approx 80$ MeV. Whether the constraint could be fulfilled requires a detailed study of the masses and their temperature corrections for the quarks and the scalar mesons, a task beyond the scope of this article. However for $T \approx T_c$ where in the strict chiral limit $m_q = 0$ and $M_{eff} \approx 80-100$ MeV, the sigma meson will be able to decay into collective excitations giving the quarks a thermal width of origin very different from the more familiar collisional width. From the one-loop expression for the scalar and pseudoscalar contributions to the quark self-energy we see that such a width will

be of order $g^2\,T$ (for a detailed expression for the width see [22]). In the perturbative analysis this width is much smaller than the effective mass of the collective excitations $\approx g\,T$.

If the temperature dependent mass of the sigma prevents its decay into collective modes by kinematics, a collisional contribution to the width of the quark appears at two loops in the quark self-energy. The argument given above is valid whenever the fermion momentum in the self-energy loop is $>> g\,T$. In particular, for $T < T_c$ but for $m_\sigma(T) >> g\,T$, the on-shell delta function that determines the width requires that the momentum of the fermion in the self-energy loop be $> g\,T$ [22] and the use of the bare quark propagator is justified. On the other hand very near the critical temperature whenever $m_\sigma(T) << g\,T$ for soft external quark momentum the internal quark line will also be soft in the kinematic region that contributes to the width. In this case a full HTL resummed quark internal line must be used. The analysis of such case has been provided in detail by Thoma [21] who concluded that the width of the quark in this limit is also given by $\approx g^2 T$. This result *includes* the collisional width but in the HTL approximation for the intermediate fermion in the self-energy diagram [21].

In summary, for scalars (and pseudoscalars) the thermal width of the quark collective modes has two different contributions both of order $g^2\,T$: the decay of the scalar (sigma) into pairs of collective modes if kinematically allowed and collisional damping in the HTL limit.

2. Gauge boson contribution:

The fermion damping rate arising from gauge boson exchange has been studied in detail in the HTL approximation by several authors [38–40]. The gauge contribution to the quark self-energy presents infrared divergences in bare perturbation theory. The infrared divergences associated with the exchange of longitudinal gauge bosons are a result of small angle Coulomb scattering and at zero temperature are those of Rutherford scattering. At finite temperature the longitudinal gauge boson (instantaneous Coulomb interaction) is screened by the Debye screening mass $m_D \propto e\,T$ which cuts off the infrared and leads to a finite contribution to the damping rate from longitudinal gauge bosons. For soft external momentum this contribution has been found to be given by [38,39]

$$\Gamma^{l,g}(k) = \alpha\ A\,T\ , \tag{2.35}$$

with A a constant that can be found numerically [38] and in QCD is of $\mathcal{O}(1)$ and $\alpha = e^2/4\pi$.

For a fermion excitation at *rest* the damping rate has been computed in [38,41,42] and found to be given by

$$\Gamma^{g}(k=0) = \alpha\ C\,T\ , \tag{2.36}$$

with C again being a numerical constant of $\mathcal{O}(1)$ [38,41].

The detailed analysis of Refs. [38,41,42] in QCD lead to the result that the damping rate of moving quasiparticles with momenta $k \gg e^2 T$ in a non-abelian plasma are given by (up to an overall constant that depends on the gauge group structure)

$$\Gamma^{g}_{\pm}(k) = \frac{e^2\, T}{4\pi} |v_{\pm}(k)| \ \log\frac{1}{e} \ , \tag{2.37}$$

where $v_{\pm}(k)$ are the group velocities for the two different branches of collective modes. This expression is *not* valid for $k = 0$ where the damping rates of the fermion and plasmino at rest do not have the logarithmic behavior in terms of the gauge coupling [38,41], nor near $k = k_{min}$ where the group velocity of the plasmino branch vanishes. Since the HTL structure in QED is similar (up to gauge group factors) to that of QCD (and scalar QED), Pisarski [38] suggested that a similar form of the damping rate should be valid in a QED plasma, despite the fact that there is no magnetic screening in the Abelian theory.

In QED the transverse photon propagator is only *dynamically* screened via Landau damping and the infrared divergences remain, possibly to all orders in perturbation theory. More recently [39] a detailed study of the fermion propagator in the Bloch-Nordsieck (eikonal) approximation in *real time* revealed that for $\omega_D\, t\, |v_{\pm}(k)| \gg 1$

$$S_k(t) \approx e^{-\alpha T |v_{\pm}(k)|\, t \ln(\omega_D\, t\, |v_{\pm}(k)|)} \ , \tag{2.38}$$

with $\omega_D \approx eT$ the Debye frequency and $v_{\pm}(k)$ the group velocity of the fermionic collective modes. Although this is not an exponential relaxation that would emerge from a Breit-Wigner resonance shape of the spectral density, it does reveal a particular time scale from which a damping rate can be extracted and is given by

$$\Gamma^{g}_{\pm}(k) \approx \alpha\, T\, |v_{\pm}(k)| \ln\frac{1}{e} \ . \tag{2.39}$$

This result has been recently found in scalar QED (which has the same HTL structure as QED and QCD to lowest order) within the dynamical renormalization group method [40].

At this stage it is important to describe the physics that leads to the damping rate from the gauge boson contribution [38–40], which is rather different from that of scalars found before. The constraints from energy momentum conservation in the imaginary part of the self-energy can only be satisfied below the light cone where the HTL resummed gauge boson propagator has support that arises from the Landau damping cut. The infrared process that leads to the non-exponential relaxation is the emission and absorption of soft photons at almost right angles with the moving fermion [38,39].

Thus we summarize the contribution to the damping rate of the collective excitations by the HTL resummed gauge boson exchange:

$$\Gamma^{g}_{\pm}(k) \approx \alpha\, T\, |v_{\pm}(k)| \ \ln\frac{1}{e} \quad \text{for} \quad k \gg g^2\, T \ , \tag{2.40}$$

$$\Gamma^{g}_{\pm}(k) \approx \alpha\, T \qquad\qquad\qquad \text{for} \quad k = 0 \ . \tag{2.41}$$

The HTL resummed retarded and advanced propagators for the soft quarks can be summarized as follows (here we also include the gauge contribution to the HTL self-energy)

$$S_R(k_0,\vec{k}) = \frac{\gamma^0\, A - \vec{\gamma}\cdot\hat{k}\; B}{A^2 - B^2}\ , \tag{2.42}$$

$$S_A(k_0,\vec{k}) = \frac{\gamma^0\, A^* - \vec{\gamma}\cdot\hat{k}\; B^*}{A^{*2} - B^{*2}}\quad , \tag{2.43}$$

$$A \pm B \equiv k_0 \pm k - i\epsilon - \frac{(g^2+e^2)\,T^2}{8k}\left[\mp 1 \pm \frac{k_0 \pm k}{2k}\ln\frac{k_0+k-i\epsilon}{k_0-k-i\epsilon}\right] + \Pi_\pm(k_0,\vec{k})\ . \tag{2.44}$$

where $\Pi_\pm(k_0,\vec{k})$ are the $\mathcal{O}\left(g^2T, \alpha T, \alpha T \ln\frac{1}{e}\right)$ corrections to the HTL self-energy that determine the width of the quark. The retarded and advanced propagators uniquely determine the time-ordered propagator via the spectral representation:

$$S^{++} = S_R + S_A + [1 - 2n_f(|k_0|)]\ \mathrm{sgn}(k_0)\ (S_R - S_A)\quad . \tag{2.45}$$

The anomalous vertex with dressed fermion propagators is shown in Fig. 3. Using the general expression (2.13), incorporating the on-shell width $\sim g^2\,T$ into our definition for A and assuming that the external momenta are negligible compared to all other scales in the problem we find that the $\pi\ A^+_\mu\ A^+_\nu$ vertex is given by:

$$S^{+++}_{\pi\gamma\gamma} \approx 64 m_q\, e^2 g \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} \times \Big\{ i\ \epsilon^{\alpha\beta\mu\nu}\ p_{1\alpha}\ p_{2\beta}\ \tilde{\pi}^+(-p_1-p_2)\ \tilde{A}^+_\mu(p_1)\ \tilde{A}^+_\nu(p_2) \tag{2.46}$$
$$\int \frac{d^4k}{(2\pi)^4}\left[\mathrm{Re}\frac{1}{A^2(k_0,\vec{k}) - B^2(k_0,\vec{k})} + [1-2n_f(|k_0|)]\mathrm{sgn}(k_0)\mathrm{Im}\frac{1}{A^2(k_0,\vec{k}) - B^2(k_0,\vec{k})}\right]^3\Bigg\}$$

The infrared region for the real part is cut-off at the scale $(g^2+e^2)T$ where we have included the HTL contribution from the gauge fields, and the imaginary part now has support below the light cone arising from the HTL and also near the mass shell of the collective modes because of the width $\Gamma \approx (g^2\,T, \alpha T, \alpha T \ln\frac{1}{e})$ which now cuts off the pinch singularity. The real time vertex is now infrared finite and free of pinch singularities and the local limit of zero frequency-momentum in the quark loop can be taken safely.

At this point we must make an important comment – in considering contributions beyond the leading order in the HTL program, we must also include consistently the vertex corrections. There are in fact two sources of vertex corrections coming from σ, π and photon exchange. As long as $g >> e$, it is the former that is more important. The inclusion of the photon exchange is necessary as a matter of principle to ensure that the Ward identities are satisfied and gauge invariance is respected. Including the vertex correction along with the

quasiparticle width makes a detailed evaluation of the effective vertex an extremely difficult task, certainly beyond the scope of this article. The conclusions that we will obtain are therefore tentative pending inclusion of the vertex corrections.

As mentioned in the introduction one of our principal goals is to understand the effective interaction of the neutral pion with gauge fields to assess the possibility of enhancement of neutral pion condensates in the presence of *classical* electromagnetic fields produced in peripheral collisions. To obtain the interaction of the pseudoscalar with classical background fields we simply replace the fluctuating fields in Eq. (2.13) with c-number expectation values or classical sources. This yields the following coupling of the pion to *classical fields* $\tilde{A}_\mu$,

$$S_{\pi\gamma\gamma} = -2e^2 g \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \tilde{\pi}^+(-p_1-p_2)\ \tilde{A}_\mu(p_1)\ \tilde{A}_\nu(p_2) \int \frac{d^4k}{(2\pi)^4} \times \tag{2.47}$$

$$\Big\{ \mathrm{Tr}[S^{++}(k-p_1)\gamma_5 S^{++}(k+p_2)\gamma^\mu S^{++}(k)\gamma^\nu] - \mathrm{Tr}[S^{-+}(k-p_1)\gamma_5 S^{++}(k+p_2)\gamma^\mu S^{+-}(k)\gamma^\nu]$$
$$- \mathrm{Tr}[S^{++}(k-p_1)\gamma_5 S^{+-}(k+p_2)\gamma^\mu S^{-+}(k)\gamma^\nu] + \mathrm{Tr}[S^{-+}(k-p_1)\gamma_5 S^{+-}(k+p_2)\gamma^\mu S^{--}(k)\gamma^\nu]\Big\}$$

Thus the interaction of the pion with classical gauge fields is determined by a combination of all the different terms in the CTP effective action. Rewriting this expression in terms of the propagators in the Keldysh notation [eqs.(2.15)-(2.17)], we obtain

$$S_{\pi\gamma\gamma} = -2e^2 g \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \tilde{\pi}^+(-p_1-p_2)\tilde{A}_\mu(p_1)\tilde{A}_\nu(p_2) \int \frac{d^4k}{(2\pi)^4} \tag{2.48}$$

$$\{\mathrm{Tr}[S_R(k-p_1)\gamma_5 S_R(k+p_2)\gamma^\mu S_R(k)\gamma^\nu] + \mathrm{Tr}[S_A(k-p_1)\gamma_5 S_A(k+p_2)\gamma^\mu S_A(k)\gamma^\nu] +$$

$$+\mathrm{Tr}[S_A(k-p_1)\gamma_5 S_F(k+p_2)\gamma^\mu S_A(k)\gamma^\nu] + \mathrm{Tr}[S_A(k-p_1)\gamma_5 S_R(k+p_2)\gamma^\mu S_F(k)\gamma^\nu] +$$

$$+\mathrm{Tr}[S_F(k-p_1)\gamma_5 S_R(k+p_2)\gamma^\mu S_R(k)\gamma^\nu]\}$$

The local limit of the pion-photon vertex is obtained by taking the external frequency and momentum to vanish in the quark loop. After the HTL resummation and accounting for the width of the intermediate quarks, this limit is unambiguous and finite. Thus we find

$$S_{\pi\gamma\gamma} =$$
$$16\, m_q\, e^2 g \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \times \Big\{ i\, \epsilon^{\alpha\beta\mu\nu}\ p_{1\alpha} p_{2\beta}\ \tilde{\pi}^+(-p_1-p_2)\tilde{A}^+_\mu(p_1)\tilde{A}^+_\nu(p_2) \int \frac{d^4k}{(2\pi)^4} \times$$

$$\times \left[\frac{1}{(A^2-B^2)^3}+\frac{1}{(A^{*2}-B^{*2})^3}+[1-2n_f(|k_0|)]\,\mathrm{sgn}(k_0)\ \left[\frac{1}{(A^2-B^2)^3}-\frac{1}{(A^{*2}-B^{*2})^3}\right]\right]\Bigg\}$$
$$+\text{vertex corrections} \tag{2.49}$$

From this result the strength of the local pion-photon vertex can be extracted, which is free of ambiguities and non-singular. It is of the form

$$g_{\pi\gamma\gamma} = \frac{m_q}{T^2}\, e^2\, g\, \mathcal{F}\left[g, \alpha, \alpha \log\frac{1}{e}, \cdots\right] \tag{2.50}$$

The function $\mathcal{F}$ is finite and dimensionless and certainly quite complicated by the logarithmic dependence on the coupling of the width and by the vertex correction. It is beyond the scope of this article to obtain a closed or approximate expression for this function. The main point of writing the eqn. (2.50) is to emphasize that after curing the pinch singularities by including the width (and vertex corrections) the vertex vanishes at $T = T_c$ in the strict chiral limit. This is in agreement with the results in [13]. The necessity of including HTL corrections was pointed out by [17]. In the present context, the CTP effective action requires the inclusion of a width as well, in order to regulate the pinch singularities. This is one of the important results of the present work.

A crude, order of magnitude estimate for the vertex may be found in the narrow-width approximation, assuming that the width $\Gamma \sim O(g^2T)$. Then, the dominant temperature dependent term in $g_{\pi\gamma\gamma}$ is $\sim e^2 g m_q \times (T^2/\Gamma^4)$ which is $O(e^2/(g^7T^2))$. (The $T^2/(\Gamma^4)$ factor may be easily understood by going back to the discussion after Eq. (2.23).)

Before we embark on the study of nonequilibrium effects it is convenient to summarize the conclusions for the equilibrium case

- The real time interaction vertices for the pseudoscalar and photons obtained by integrating out the quarks in the real time formulation display infrared divergences in the local limit when the bare quark propagators are used with $m_q = 0$. These divergences are cured by an HTL resummation of the quark lines *and* by considering contributions beyond HTL that give an on-shell width to the collective excitations. The resummed quark propagators now give rise to an unambiguous and finite local interaction vertex.

- The computation of the effective vertex in *imaginary time* in terms of bare propagators [13,14,31] cannot be used to discuss either the decay amplitude of the neutral pion or any dynamical aspects. In that computation the external Matsubara frequencies were set to zero and therefore the analytic continuation necessary to obtain the real time vertices cannot be performed. This computation cannot reveal the subtle infrared and pinch singularities that arise from finite temperature contributions to the discontinuities of the triangle diagram that are responsible for ambiguities in the limit of soft external momenta. A resummation of the quark propagator including both HTL contributions *and* the corrections that lead to the width of the fermionic collective modes is necessary to regulate the infrared divergences in the soft limit.

- The infrared and pinch singularities of the real time vertex imply that the local limit of the pseudoscalar-photon vertex is very sensitive to the details of long-wavelength physics and is therefore *non-universal* in the sense that it depends on the details of the underlying model. This is in striking contrast with the situation in the vacuum where the vertex in the local limit is completely determined by the triangle anomaly and the Adler-Bardeen 'theorem' [43] guarantees that the vertex does not receive radiative corrections. At finite temperature the situation is more complicated. As originally observed in [14], while the anomaly equation is independent of temperature and does not receive radiative corrections, the vertex itself is not determined solely by the anomaly. Our studies in equilibrium complement those of Gelis [17] and lead to the conclusion that the vertex is sensitive to the infrared and thus to the details of the low energy scalar and pseudoscalar sectors.

- After the HTL resummation and accounting for higher order corrections that determine the fermion width the local limit of the vertex is given by Eq. (2.50) which vanishes in the strict chiral limit $m_q = 0$ in the symmetric phase, confirming the results in [13,14]. Hence, despite the fact that the imaginary time computation of the vertex had missed the infrared sensitivity, it does reveal the *correct* behavior for the pseudoscalar-photon vertex in the strict chiral limit near the phase transition, when the constituent quark mass vanishes.

The full benefits of the real-time effective action will be appreciated in a truly out of equilibrium situation. In the earlier sections we reproduced previous *equilibrium* results within this approach and pointed out certain important subtleties pertaining to the appearance of pinch singularities. We now turn our attention to the true non-equilibrium situation which is the focus of our work.

III. $\pi\gamma\gamma$ OUT OF EQUILIBRIUM

We expect that nonequilibrium phenomena will be the rule rather than the exception in heavy ion collisions. In particular, in terms of DCC formation, we might expect that since electromagnetic effects break isospin, they might be able to bias the neutral to charged pion ratios in a way that could be detected in an event by event analysis.

The scenarios of formation of DCC-type configurations envisage a strongly 'quenched' phase transition, in which the cooling of the plasma occurs via hydrodynamic expansion on a time scale far shorter than the equilibration scale of long-wavelength fluctuations [6–9]. Within the setting of the linear sigma model this situation is modeled by suddenly changing the form of the potential for the scalars (and pseudoscalars) from one with an unbroken symmetry minimum to another that displays symmetry breaking minima.

While most of the previous studies focussed on an effective theory of scalars and pseudoscalars (mainly the linear sigma model), in order to study the effects of the anomalous vertex on possible enhancements of neutral pion condensates (constituent) quarks must be

added to the model. Following phenomenologically motivated descriptions [13,14] we adopt the $U(2) \otimes U(2)$ gauged constituent quark model. This is, of course, a simple extension of the model of the last section, with an $SU(2)_L \times SU(2)_R$ global symmetry with two quark flavours and $N_c = 3$ colors, coupled to electromagnetism:

$$\mathcal{L} = i\bar{\psi}(\not{\partial} - ieQ\not{A})\psi - 2g\bar{\psi}(\sigma\tau^0 + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_\pi. \tag{3.1}$$

Here $(\tau^0, \vec{\tau})$ are the isospin generators proportional to the Pauli matrices, satisfying the normalization conditions $\mathrm{Tr}(\tau^a\tau^b) = \delta^{ab}/2$, $\tau^0 = \mathbf{1}/2$ and

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

where we have suppressed all color indices. The $U(1)_{em}$ charge is given by $Q = \tau^0/3 + \tau^3$. $\mathcal{L}_\pi$ determines the dynamics of the pions and the sigma field and is a gauged $O(4)$ scalar theory given by,

$$\mathcal{L}_\pi = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{e^2}{2}(\pi_1^2 + \pi_2^2)A_\mu A^\mu + j_\mu A^\mu - \lambda(\sigma^2 + \vec{\pi}^2 - v_0)^2 + h\sigma \tag{3.2}$$

$$j_\mu = e\,(\pi_1\partial_\mu\pi_2 - \pi_2\partial_\mu\pi_1) \tag{3.3}$$

Our approach to the problem will be to consider the evolution in time of the chiral order parameter/condensate $\langle\sigma\rangle$, (where the expectation value is taken in the time evolving density matrix), and to study how this evolution feeds back into the fermion propagators in the triangle diagram.

We consider the initial state to be described by free fields in thermal equilibrium at a temperature $T_i > T_c$ and evolve the equations of motion with the $T = 0$ masses. This 'quench' scenario was originally described in [8,27]. Several authors studied the cooling by hydrodynamic expansion [6,7,9] with similar results. Therefore we consider preparing an initial density matrix that describes the system in equilibrium at an initial temperature above the critical at an initial time t_i which for convenience will eventually be taken to be $t_i = 0$. This initial density matrix is now evolved in time with a potential corresponding to the zero temperature situation [27].

At time $t = t_i$ the temperature suddenly falls below the critical value and the potential develops two (non-degenerate) minima. The condensate now evolves in time from the initial value towards the final equilibrium point with the minimum (free) energy. We always begin with an equilibrium initial state where the chiral order parameter has a non-zero expectation value $\langle\sigma(x)\rangle \propto h$ which is the explicit symmetry breaking term and yields a non-zero bare mass for the quarks.

We should emphasize at the outset that the infrared and pinch singularities that plagued the equilibrium calculation are *absent* in this scenario. The reasons for this are

1. We begin with a non-zero condensate value which yields a finite constituent quark mass at $T_i > T_c$. This arises from the explicit symmetry breaking in the linear sigma model, which is responsible for the physical masses of the pions.

2. The range of integration time is *finite*, from an initial time $t_i = 0$ up to the time t of interest when the phase transition is complete, $t \approx 10$ fm/c. The fact that the time interval is finite introduces a natural infrared cutoff and prevents the pinch singularities associated with the propagation of on-shell intermediate states over very long times as emphasized in the non - equilibrium situation [44]. That real time acts as an infrared cutoff for the propagation of intermediate states has been understood within the framework of the damping of collective excitations in a plasma in [39,40].

Since the scalar couples to quarks and pions, the rolling of $\langle\sigma(x)\rangle = \sigma_0(t)$ has two important effects: i) an effective *time dependent* mass for the pion fields that leads to instabilities in the spinodal region of the potential and consequent growth of pion fluctuations [6–9,27], ii) a time dependent constituent mass for the quarks.

These nonequilibrium effects result in the production of a large number of pions via spinodal growth of pion fluctuations [6–9,27,29] and production of $q\bar{q}$ pairs through the time dependence of their mass [28]. The quantum backreaction of quarks and pions affects the dynamics of $\langle\sigma(x)\rangle$ so that we cannot simply focus on the classical equations of motion and a self-consistent treatment of the dynamics of the condensate incorporating quantum effects is essential. Detailed investigations of such dynamical situations in the $O(N)$ model and the scalar sector of the linear sigma model using nonperturbative schemes such as the large-N limit have been carried out in previous works by various groups [7,9,27,29] without the inclusion of quarks. The out of equilibrium evolution including fermion fields is treated in [28].

As before, the full quantum field $\sigma^{\pm}(x)$ can be split up into its classical, c-number expectation value and quantum fluctuations about such expectation value

$$\begin{aligned} &\sigma^{\pm}(\vec{x},t) = \sigma_0(t) + \chi^{\pm}(\vec{x},t) \\ &\langle\sigma^{\pm}(\vec{x},t)\rangle = \sigma_0(t) \quad , \quad \langle\chi^{\pm}\rangle = 0 \; . \end{aligned} \tag{3.4}$$

The condensate will be introduced as a time-dependent mass term for the quarks and its dynamics will be treated non-perturbatively. We split the fermionic part of the Lagrangian into a free part and interaction terms as follows:

$$\begin{aligned} &\mathcal{L}_f = \mathcal{L}_0 + \mathcal{L}_{int} \\ &\mathcal{L}_0 = i\bar{\psi}^{+}[i\partial\!\!\!/ - g\sigma_0(t)]\psi^{+} - (+ \longrightarrow -) \\ &\mathcal{L}_{int} = e\bar{\psi}^{+}\gamma^{\mu}QA^{+}_{\mu}\psi^{+} - ig\vec{\tau}\cdot\vec{\pi}^{+}\bar{\psi}^{+}\gamma_5\psi^{+} - (+ \longrightarrow -) + \cdots . \end{aligned} \tag{3.5}$$

where the dots stand for terms which are not relevant to the pseudoscalar-photon vertex to one loop in the triangle diagram. We now combine the results of the (mean-field) large N

equations for the linear sigma model [6,7,9,27,29] with those obtained for the Yukawa theory out of equilibrium [28] to obtain the self-consistent equations of motion in the *mean field* approximation.

$$
\begin{aligned}
&\ddot{\sigma}_0(t) - 4\lambda\, v_0^2\, \sigma_0(t) + 4\lambda\, \sigma_0^3(t) + 4\lambda\, \sigma_0(t)\, \langle \vec{\pi}^2(x)\rangle - g\langle \bar{\psi}\psi\rangle = h \\
&\langle \vec{\pi}^2(x)\rangle = \frac{3}{2}\int \frac{d^3k}{(2\pi)^3}\, |\Phi_k(t)|^2 \coth\left[\frac{\omega_{k,\pi}(0)}{2T_i}\right] \\
&\ddot{\Phi}_k(t) + \omega_{k,\pi}^2(t)\ \Phi_k(t) = 0 \ \ ; \ \ \omega_{k,\pi}^2(t) = k^2 - 4\lambda\ v_0^2 + 4\lambda\ \sigma_0^2(t) + 4\lambda\ \langle \vec{\pi}^2(x)\rangle \\
&[i\partial\!\!\!/ - g\sigma_0(t)]\psi(x) = 0
\end{aligned}
\tag{3.6}
$$

where we have set the number of pions $N = 3$. The reader is referred to [6,7,9,27–29] for details on these equations and their initial conditions.

In the above equations we have neglected the contribution from the classical gauge fields to the back-reaction problem for the dynamics of the condensate. This is justified since the electromagnetic effects are of $\mathcal{O}(e^2)$ but $\lambda, g >> e$ and $\lambda \approx g$ [6,7,9,27,29]. This set of coupled self-consistent equations determine the time evolution of the condensate including the large contributions from the pion and fermionic fluctuations. Although the integrals associated with $\langle\vec{\pi}^2\rangle$; $\langle\bar{\psi}\psi\rangle$ are ultraviolet divergent and renormalization would be required, the model under study is only useful as a low energy effective description with a natural cutoff $\Lambda \leq 1$ GeV.

The fermion propagators in the background of the condensate for a 'quench' scenario from an initial state with temperature T_i are given by [28,45]:

$$S^{+-}_{\vec{k}}(t,t') = -S^{<}_{\vec{k}}(t,t') = -i\int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ \langle \bar{\psi}^-(\vec{0},t')\psi^+(\vec{x},t)\rangle\ , \tag{3.7}$$

$$S^{-+}_{\vec{k}}(t,t') = -S^{>}_{\vec{k}}(t,t') = i\int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ \langle \psi^-(\vec{x},t)\bar{\psi}^+(\vec{0},t')\rangle\ , \tag{3.8}$$

$$S^{++}_{\vec{k}}(t,t') = S^{>}_{\vec{k}}(t,t')\Theta(t-t') + S^{<}_{\vec{k}}(t,t')\Theta(t'-t)\ , \tag{3.9}$$

$$S^{--}_{\vec{k}}(t,t') = S^{>}_{\vec{k}}(t,t')\Theta(t'-t) + S^{<}_{\vec{k}}(t,t')\Theta(t-t')\ , \tag{3.10}$$

$$S^{>}_{\vec{k}}(t,t') = S^{>}_{0\vec{k}}(t,t')\,[1-n_f(\omega_k)] - S^{<}_{0\vec{k}}(t,t')n_f(\omega_k)\ , \tag{3.11}$$

$$S^{<}_{\vec{k}}(t,t') = S^{<}_{0\vec{k}}(t,t')\,[1-n_f(\omega_k)] - S^{>}_{0\vec{k}}(t,t')n_f(\omega_k)\ , \tag{3.12}$$

$$\omega_k = \sqrt{\vec{k}^2 + g^2\sigma^2(0)}\ , \qquad\qquad n_f(\omega_k) = \frac{1}{e^{\beta\omega_k}+1}\ . \tag{3.13}$$

We have listed the nonequilibrium fermion propagators for the case where the initial state is thermal with temperature $T_i = 1/\beta$ and the chiral condensate has the initial value $\sigma(0)$ for all times $t < t_i$. $S^{>}_{0\vec{k}}$ and $S^{<}_{0\vec{k}}$ are the Wightman functions for the zero temperature case and are given by

$$
\begin{aligned}
&S^{>}_{0k}(t,t') = \\
&-if_k(t)f^*_k(t')\left[\mathcal{W}_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} + g\sigma_0(t)\right]\left(\frac{1+\gamma_0}{2}\right)\left[\mathcal{W}^*_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} + g\sigma_0(t')\right] \; , \\
&S^{<}_{0k}(t,t') = \\
&-if^*_k(t)f_k(t')\left[\mathcal{W}^*_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} - g\sigma_0(t)\right]\left(\frac{1-\gamma_0}{2}\right)\left[\mathcal{W}_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} - g\sigma_0(t')\right] \; ,
\end{aligned}
\tag{3.14}
$$

$$
\mathcal{W}_k(t) = i\,\frac{\dot{f}_k(t)}{f_k(t)} \; , \tag{3.15}
$$

$$
\left[\frac{d^2}{dt^2} + k^2 + g^2\sigma_0^2(t) - ig\dot{\sigma}_0(t)\right] f_k(t) = 0 \; , \tag{3.16}
$$

$$
f_k(t < t_i = 0) = \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k(\omega_k + g\sigma_0(t_i = 0))}} \; . \tag{3.17}
$$

where the mode functions $f_k(t)$ determine the solution of the time dependent Dirac equation. Details of the solution to the Dirac equation and the derivation of the nonequilibrium propagators for the fermions are provided in Appendix B. It is this dressing of the fermions by the time-dependent condensate which will ultimately lead to nonequilibrium modifications of the $\pi^0\gamma\gamma$ couplings. In terms of these propagators we find [28]

$$
\langle\bar{\psi}\psi\rangle = \int \frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}S^{<}_{k}(t;t) = 2N_fN_c\int\frac{d^3k}{(2\pi)^3}\left[1 - 2k^2|f_k(t)|^2\right] \tag{3.18}
$$

with N_f, N_c the number of flavors (2) and of colors (3) respectively. As mentioned in the introduction, it is not our goal here to fully study the coupled set of equations, but to focus on how this mean field evolution provides nonequilibrium effects that modify the pseudoscalar-photon coupling.

A. Integrating out the fermions:

As in the equilibrium case, the effective couplings will be induced once the fermions are integrated out at the one-loop level via the triangle diagram. As before, we extract the nonequilibrium couplings by expanding $\exp[iS_{int}]$ and integrating out the fermions at one-loop. This procedure automatically incorporates all the effects of the medium or the condensate which couples to the fermions in the loops and we find the following expression for the $\pi^0\gamma\gamma$ interactions in the nonequilibrium effective action:

$$
S^{noneq}_{\pi\gamma\gamma} = -\frac{ge^2 N_c}{3} \int d^4x \; d^4x_1 \; d^4x_2 \times
$$

$$
\times \Big\{ \pi^{0+}(x) A^+_\mu(x_1) A^+_\nu(x_2) \mathrm{Tr}[S^{++}(x_2,x)\gamma_5 S^{++}(x,x_1)\gamma^\mu S^{++}(x_1,x_2)\gamma^\nu] \tag{3.19}
$$

$$
-\pi^{0+}(x) A^+_\mu(x_1) A^-_\nu(x_2) \mathrm{Tr}[S^{>}(x_2,x)\gamma_5 S^{++}(x,x_1)\gamma^\mu S^{<}(x_1,x_2)\gamma^\nu]
$$

$$
-\pi^{0+}(x) A^-_\mu(x_1) A^+_\nu(x_2) \mathrm{Tr}[S^{++}(x_2,x)\gamma_5 S^{<}(x,x_2)\gamma^\nu S^{>}(x_2,x_1)\gamma^\mu]
$$

$$
+\pi^{0+}(x) A^-_\mu(x_1) A^-_\nu(x_2) \mathrm{Tr}[S^{>}(x_2,x)\gamma_5 S^{<}(x,x_2)\gamma^\nu S^{--}(x_2,x_1)\gamma^\mu] \Big\} \, .
$$

where $S^>, S^<, S^{++}$ are the nonequilibrium fermionic propagators given by (3.7-3.17) (see Appendix B for details).

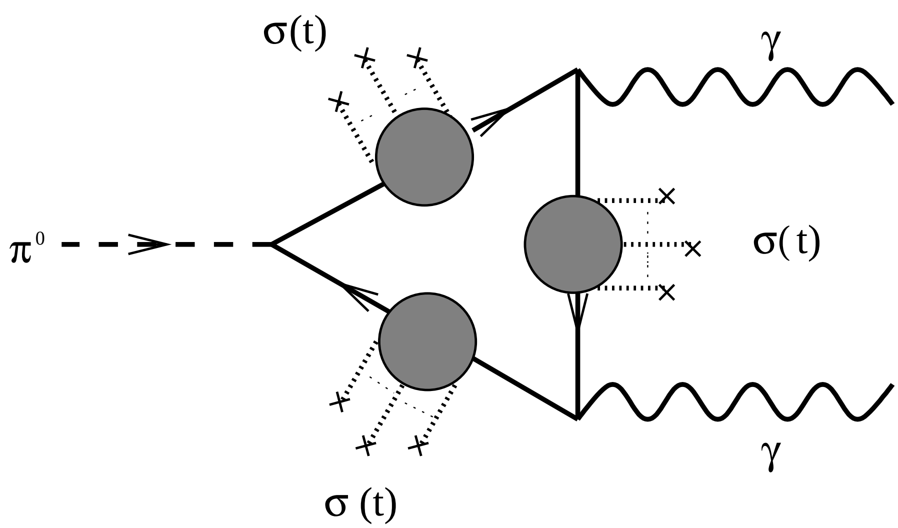

FIG.4: The $\pi\gamma\gamma$ vertex out of equilibrium.

Fig. 4, illustrates the nature of the anomalous couplings which are induced out of equilibrium. Note that cross-couplings between fields living on different branches of the CTP contour have been induced in Eq. (3.19). We note that if the zero temperature, equilibrium form for the fermion propagators is used the cross-couplings vanish and only the $\pi^{0+}A^+A^+$ vertex remains. Now the effective interactions in Eq. (3.19) are manifestly non-local. In practice a local vertex is obtained via a derivative expansion or an expansion in powers of

the external 4-momenta where each derivative is suppressed by the mass of the fermions in the loop which are being decoupled. In the usual equilibrium case the fermion mass is simply $gv_0 \approx gf_\pi$ and the derivative expansion is an expansion in powers of (∂_μ/gf_π) or an expansion in powers of (∂_μ/T) in the high temperature regime.

In the strongly out-of-equilibrium scenario under study, such a derivative expansion is unwarranted. Furthermore, the lack of time translational invariance due to the rolling of the condensate out of equilibrium prevents Fourier transforms in time and a subsequent expansion in frequencies. The Green's functions and consequently the kernels are no longer functions of time differences, but in fact functions of the individual time arguments. It is also clear that separating the center of mass and relative time variables as in the case of a Wigner representation is of little use since there is no obvious separation of time scales.

However if the electromagnetic field can be treated *classically* and if it is slowly varying on the time scale of evolution of the condensate ($5-10$ fm/c [6,7,9,27]) we can extract a local interaction term of the type $\sim g_{\pi\gamma\gamma}(t)\ \pi^0\ \vec{E}\cdot\vec{B}$ where the time-dependent coefficient determines the strength of the anomalous coupling during the stage of nonequilibrium evolution of the condensate. The restriction to static classical electromagnetic fields is consistent with the assumptions used by [20] in their estimate of anomalous enhancement of neutral pion condensates.

B. Coupling to quasi-static $\vec{E}$ and $\vec{B}$ fields:

Although the non-localities in the nonequilibrium vertex are difficult to handle in general, couplings to quasi-static electromagnetic fields are relatively easy to obtain and we will now outline the derivation of these couplings. The assumption of quasi-static electromagnetic fields entails that their time variation is on scales much longer than those of the nonequilibrium kernels ($\approx 5-10$fm/c [6,7,9,27]) that enter in the vertex. Under this assumption we can take these electromagnetic fields outside of the time integral. We first introduce spatially Fourier-transformed fields:

$$\begin{aligned}\tilde{\pi}^{0+}(\vec{q},t) &= \int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\ \pi^{0+}(\vec{x},t)\ ,\\ \tilde{A}^{\pm}_\mu(\vec{q},t) &= \int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\ A^{\pm}_\mu(\vec{x},t)\ .\end{aligned} \tag{3.20}$$

In terms of these fields, we find that the following anomalous interactions are induced at one-loop in the nonequilibrium action:

$$S^{noneq}_{\pi\gamma\gamma} = -e^2 g\frac{N_c}{3}\int dt dt_1 dt_2 \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\times \tag{3.21}$$
$$\times\Bigg[\tilde{\pi}^{0+}(\vec{q}_1,t)\tilde{A}^{+}_\mu(\vec{q}_2,t_1)\tilde{A}^{+}_\nu(-\vec{q}_1-\vec{q}_2;t_2)\int\frac{d^3k}{(2\pi)^3}V^{\mu\nu}_{+++}(\vec{q}_1+\vec{q}_2+\vec{k},\vec{k}+\vec{q}_2,\vec{k};t,t_1,t_2)+$$

$$-\tilde{\pi}^{0+}(\vec{q}_1,t)\tilde{A}^{+}_{\mu}(\vec{q}_2,t_1)\tilde{A}^{-}_{\nu}(-\vec{q}_1-\vec{q}_2;t_2)\int\frac{d^3k}{(2\pi)^3}V^{\mu\nu}_{++-}(\vec{q}_1+\vec{q}_2+\vec{k},\vec{k}+\vec{q}_2,\vec{k};t,t_1,t_2)+$$

$$-\tilde{\pi}^{0+}(\vec{q}_1,t)\tilde{A}^{-}_{\mu}(\vec{q}_2,t_1)\tilde{A}^{+}_{\nu}(-\vec{q}_1-\vec{q}_2;t_2)\int\frac{d^3k}{(2\pi)^3}V^{\mu\nu}_{+-+}(\vec{q}_1+\vec{q}_2+\vec{k},\vec{k}+\vec{q}_2,\vec{k};t,t_1,t_2)+$$

$$+\tilde{\pi}^{0+}(\vec{q}_1,t)\tilde{A}^{-}_{\mu}(\vec{q}_2,t_1)\tilde{A}^{-}_{\nu}(-\vec{q}_1-\vec{q}_2;t_2)\int\frac{d^3k}{(2\pi)^3}V^{\mu\nu}_{+--}(\vec{q}_1+\vec{q}_2+\vec{k},\vec{k}+\vec{q}_2,\vec{k};t,t_1,t_2)\Big]$$

where, $V^{\mu\nu}_{+++}$, $V^{\mu\nu}_{++-}$, $V^{\mu\nu}_{+-+}$, and $V^{\mu\nu}_{+--}$ are given by the following expressions in terms of the nonequilibrium propagators,

$$V^{\mu\nu}_{+++}(\vec{k}_1,\vec{k}_2,\vec{k}) = \tag{3.22}$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^{\mu}S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_2-t)\Theta(t-t_1)\Theta(t_2-t_1)+$$

$$\mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu}S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t-t_2)\Theta(t_1-t)\Theta(t_1-t_2)+$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu}S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_2-t)\Theta(t_1-t)\Theta(t_1-t_2)+$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu}S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_2-t)\Theta(t_1-t)\Theta(t_2-t_1)+$$

$$\mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^{\mu}S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t-t_2)\Theta(t-t_1)\Theta(t_1-t_2)+$$

$$\mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^{\mu}S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t-t_2)\Theta(t-t_1)\Theta(t_2-t_1)\ ,$$

$$V^{\mu\nu}_{++-}(\vec{k}_1,\vec{k}_2,\vec{k}) = \tag{3.23}$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^{\mu}S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t-t_1)+$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu}S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_1-t)\ ,$$

$$V^{\mu\nu}_{+-+}(\vec{k}_1,\vec{k}_2,\vec{k}) = \tag{3.24}$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu}S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_2-t)+$$

$$\mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu} S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t-t_2)\ ,$$

$$V^{\mu\nu}_{+--}(\vec{k}_1,\vec{k}_2,\vec{k}) = \tag{3.25}$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu} S^{>}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_2-t_1)+$$

$$\mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^{\mu} S^{<}_{k}(t_1,t_2)\gamma^{\nu}]\Theta(t_1-t_2)\ ,$$

and $\vec{k}_1 = \vec{k} + \vec{q}_1 + \vec{q}_2,\ \vec{k}_2 = \vec{k} + \vec{q}_2$.

We now assume that the photons can be treated as classical quasi-static background fields in accord with our original goal of studying the response of the neutral pion field to large electromagnetic fields in peripheral collisions and in agreement with the treatment of [20,9]. We then perform a spatial derivative expansion to extract a local coupling. The result will be gauge-invariant under restricted (static) gauge transformations. While such couplings cannot be used to obtain the pion *width* these are relevant for addressing the possibility of neutral pion condensates induced by the electromagnetic fields produced in peripheral collisions or through the phase transition.

We introduce c-number expectation values (mean fields) for the electromagnetic fields,

$$A^{classical}_{\mu} = \langle A^{\pm}_{\mu} \rangle = \mathcal{A}_{\mu}. \tag{3.26}$$

In the presence of these classical sources the coupling of π^0 to the electromagnetic fields takes the following form,

$$\begin{aligned}
&-e^2 g\frac{N_c}{3}\int_{t_i}^{\infty} dt \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\ \ \tilde{\pi}^{0+}(\vec{q}_1,t)\ \tilde{\mathcal{A}}_{\mu}(\vec{q}_2)\ \tilde{\mathcal{A}}_{\mu}(-\vec{q}_1-\vec{q}_2)\ \times \\
&\times \int_{t_i}^{\infty} dt_1 \int_{t_i}^{\infty} dt_2 \int \frac{d^3k}{(2\pi)^3}(V^{\mu\nu}_{+++} - V^{\mu\nu}_{++-} - V^{\mu\nu}_{+-+} + V^{\mu\nu}_{+--}).
\end{aligned} \tag{3.27}$$

This is obtained from Eq. (3.21) by replacing the fluctuating gauge fields with their c-number expectation values. The range of the time integrals in the expressions above is from $t_i = 0$ to t as determined by the step-functions (see below).

In this nonequilibrium scenario, the anomalous coupling to classical fields is then

$$e^2 g\frac{N_c}{3}\int dt \int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\ \ \tilde{\pi}^{0+}(\vec{q}_1,t)\ \tilde{\mathcal{A}}_{\mu}(\vec{q}_2)\ \tilde{\mathcal{A}}_{\mu}(-\vec{q}_1-\vec{q}_2)\times V^{\mu\nu}(\vec{q}_1,\vec{q}_2;t). \tag{3.28}$$

where

$$V^{\mu\nu}(\vec{q}_1,\vec{q}_2;t) = \int_{t_i}^{\infty} dt_1 \int_{t_i}^{\infty} dt_2 \int \frac{d^3k}{(2\pi)^3} \times$$

$$\Big[-\Big\{ \mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^\mu S^{<}_{k}(t_1,t_2)\gamma^\nu] + \mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^\mu S^{>}_{k}(t_1,t_2)\gamma^\nu] \Big\}$$

$$\times \Theta(t-t_1)\Theta(t-t_2)+$$

$$+\Big\{ \mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^\mu S^{>}_{k}(t_1,t_2)\gamma^\nu] + \mathrm{Tr}[S^{>}_{k_1}(t_2,t)\gamma_5 S^{<}_{k_2}(t,t_1)\gamma^\mu S^{<}_{k}(t_1,t_2)\gamma^\nu] \Big\} \times$$

$$\Theta(t-t_1)\Theta(t-t_2)\Theta(t_2-t_1)+$$

$$+\Big\{ \mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^\mu S^{>}_{k}(t_1,t_2)\gamma^\nu] + \mathrm{Tr}[S^{<}_{k_1}(t_2,t)\gamma_5 S^{>}_{k_2}(t,t_1)\gamma^\mu S^{<}_{k}(t_1,t_2)\gamma^\nu] \Big\} \times$$

$$\times \Theta(t-t_2)\Theta(t-t_1)\Theta(t_2-t_1)\Big] \ ,$$

where, as noted before $\vec{k}_1 = \vec{k} + \vec{q}_1 + \vec{q}_2$ and $\vec{k}_2 = \vec{k} + \vec{q}_2$.

We now expand the above traces in powers of $\vec{q}_1$ and $\vec{q}_2$ which corresponds to performing a *spatial* derivative expansion. Furthermore since we are confining ourselves to interactions with static gauge fields it suffices to retain only those terms in the traces which yield couplings of the type $\pi^0 \epsilon^{ijk}\, \partial_i \mathcal{A}_0\, \partial_j \mathcal{A}_k$. After considerable but straightforward algebra, the details of which are not particularly illuminating we find that the local effective vertex for the neutral pion coupling to static, classical $\vec{E}$ and $\vec{B}$ fields is given by

$$S^{noneq}_{\pi\gamma\gamma} = -4 \int d^4x \;\; g_{\pi\gamma\gamma}(t)\; \epsilon^{ijk}\; \pi^{0+}(\vec{x},t)\; \partial_i \mathcal{A}_0(\vec{x})\; \partial_j \mathcal{A}_k(\vec{x}) \tag{3.29}$$

where the amplitude or strength of the time-dependent anomalous coupling is:

$$g_{\pi\gamma\gamma}(t) = -\frac{ge^2N_c}{6} \int_{t_i}^{\infty} dt_1 \int_{t_i}^{\infty} dt_2 \int \frac{d^3k}{(2\pi)^3} \times \tag{3.30}$$

$$\times \Bigg[\Bigg\{ X^{-*}_k(t_1,t_2) \;\; \left(1 + \frac{2k^2}{3}\frac{\partial}{\partial k^2}\right) \;\; [Y_k(t_2,t)\; Y_k(t,t_1) + Z_k(t_2,t)\; Z_k(t,t_1)] +$$

$$-\frac{2k^2}{3} \;\; Y^*_k(t_1,t_2) \;\; \Bigg[Y_k(t,t_1)\; \frac{\partial}{\partial k^2} X^-_k(t_2,t) + Y^*_k(t,t_2)\; \frac{\partial}{\partial k^2} X^{-*}_k(t_1,t) +$$

$$-Z_k(t,t_1)\; \frac{\partial}{\partial k^2} X^+_k(t_2,t) - Z^*_k(t,t_2)\; \frac{\partial}{\partial k^2} X^{+*}_k(t_1,t) \Bigg] \Bigg\} \;\; \Theta(t-t_1)\Theta(t-t_2)+$$

$$
+\left\{X_k^-(t_1,t_2)\;\left(1+\frac{2k^2}{3}\frac{\partial}{\partial k^2}\right)\;\left[-Y_k(t_2,t)\;Y_k^*(t,t_1)+Z_k(t_2,t)\;Z_k^*(t,t_1)\right]+\right.
$$

$$
+\frac{2k^2}{3}\;Y_k(t_1,t_2)\;\left[Y_k^*(t,t_1)\;\frac{\partial}{\partial k^2}X_k^-(t_2,t)+Y_k^*(t,t_2)\;\frac{\partial}{\partial k^2}X_k^-(t_1,t)+\right.
$$

$$
\left.\left.+Z_k^*(t,t_1)\;\frac{\partial}{\partial k^2}X_k^+(t_2,t)+Z_k^*(t,t_2)\;\frac{\partial}{\partial k^2}X_k^+(t_1,t)\right]+\text{c.c.}\right\}\times
$$

$$
\Theta(t-t_1)\Theta(t-t_2)\Theta(t_2-t_1)\big]\,.
$$

The variables $X_k^{\pm},\ Y_k,\ Z_k$ are related to the mode functions f_k via the following definitions:

$$
X_k^-(t,t')=f_k(t)f_k^*(t')\left[S_k(t)S_k^*(t')-k^2\right](1-n_f)-f_k^*(t)f_k(t')\left[S_k^*(t)S_k(t')-k^2\right]n_f, \tag{3.31}
$$

$$
X_k^+(t,t')=f_k(t)f_k^*(t')\left[S_k(t)S_k^*(t')+k^2\right](1-n_f)+f_k^*(t)f_k(t')\left[S_k^*(t)S_k(t')+k^2\right]n_f, \tag{3.32}
$$

$$
Y_k(t,t')=-f_k(t)f_k^*(t')\left[S_k(t)+S_k^*(t')\right](1-n_f)-f_k^*(t)f_k(t')\left[S_k^*(t)+S_k(t')\right]n_f, \tag{3.33}
$$

$$
Z_k(t,t')=f_k(t)f_k^*(t')\left[S_k^*(t')-S_k(t)\right](1-n_f)+f_k^*(t)f_k(t')\left[S_k(t')-S_k^*(t)\right]n_f, \tag{3.34}
$$

$$
S_k(t)=i\;\frac{\dot{f}_k(t)}{f_k(t)}+g\;\sigma_0(t). \tag{3.35}
$$

While these expressions look rather unwieldy, it is clear that the $\pi^0\gamma\gamma$ vertex receives important nonequilibrium corrections through the evolution of the condensate. In the self-consistent mean-field treatment advocated here, we see that the effective vertex out of equilibrium receives contributions from *all the field fluctuations* through the back-reaction.

These are **memory** effects since the anomalous coupling at time t depends on the fields at all times $t' \leq t$. Analogous effects are known to appear in related phenomena as in the nonequilibrium dynamics of the condensate [6,7,9,27,29,30] where pion spinodal fluctuations and the nonlinear backreaction are very important. Hence, just as in the previous study of the effective vertex in equilibrium we conclude that, unlike the anomaly equation (see Appendix A), the effective pseudoscalar-photon vertex receives important corrections from long-wavelength fluctuations and is not determined by the anomaly equation.

We are now in position to obtain the effective equations of motion for the neutral pion field including the self-consistent mean field evolution as well as the effective low energy interaction vertex with the classical electromagnetic fields:

$$\partial_\mu \partial^\mu \pi^0(\vec{x},t) + M^2(t)\; \pi^0(\vec{x},t) = -g_{\pi\gamma\gamma}(t)\; \vec{E}(\vec{x},t)\cdot \vec{B}(\vec{x},t) \tag{3.36}$$

$$M^2(t) = -4\lambda\; v_0^2 + 4\lambda\; \sigma_0^2(t) + 4\lambda\; \langle \vec{\pi}^2(x)\rangle \tag{3.37}$$

where the last term in (3.37) can be read from Eq. (3.6).

This equation displays several important physical processes: i) early time spinodal instabilities resulting from the negative contribution to the mass term, just as in the more conventional treatments of formation of DCC [6,7,9,27,29], ii) an inhomogeneity induced by the classical quasi-static electromagnetic fields as a result of the anomalous coupling but with a vertex that is a strong and rapid function of time. This vertex must be obtained self-consistently from the coupled set of mean-field equations (3.6)-(3.16) by inserting the mode functions in Eq. (3.31). Although we do not intend to offer a numerical analysis of the resulting equations in this article, the nonequilibrium equation of motion (3.36) has the potential for providing an enhancement of electromagnetic isospin breaking effects through the combination of spinodal instabilities and nonequilibrium time dependence of the pseudoscalar vertex. This enhancement *could* lead to an experimentally observable signal in the neutral to charged pion ratio. We postpone to a forthcoming article the full numerical study of these equations and an assessment of the potential phenomenological impact of the nonequilibrium dynamics.

IV. DISCUSSIONS, CONCLUSIONS AND OUTLOOK

The axial anomaly displays a wealth of fascinating phenomena at finite temperature as well as in out of equilibrium situations, perhaps even more than at zero temperature. What we have done in this work is to use the techniques of real time quantum field theory to understand the relationship between the axial anomaly and the $\pi^0 \rightarrow \gamma\gamma$ amplitude and develop a a systematic description of the in-medium equilibrium and nonequilibrium corrections to the pseudoscalar-photon vertex.

The real time equilibrium calculation allowed us to make contact with some previous results. In particular, we obtained agreement with the results in [17] by finding that the usual perturbative calculation of the triangle diagram in real time and finite temperature in terms of bare quark propagators is afflicted by infrared divergences in the strict chiral limit of vanishing quark masses, and pinch singularities in the local limit of the vertex. The latter singularities for vanishing external four momentum are a manifestation of the propagation of undamped on-shell intermediate states in the medium. Both types of divergences arise from the soft momentum region for the internal quark lines. In this region the quark propagators must be resummed including the HTL contribution from scalars and gauge bosons. The pinch singularities, however, are not cured by the HTL resummation but require going beyond the HTL limit and accounting for the on-shell width of the collective excitations. These conclusions are still somewhat tentative, since we have not included the vertex corrections whose contributions can be significant (see for e.g. [47]).

After the infrared and pinch singularities had been smoothed out by HTL resummation and the inclusion of the quasiparticle width, we found that in the strict chiral limit, the anomalous vertex vanishes when $m_q \to 0$ in agreement with the original results of Pisarski [13,14,31] and the suggestion by Gelis [17].

We next focused on the description of nonequilibrium dynamics during a quenched chiral phase transition described by a constituent quark model with a scalar-pseudoscalar sector described by the linear sigma model. The dynamics of a quenched chiral phase transition is studied in a self-consistent mean-field theory as advocated in many treatments of DCC formation [6–9,27] but including the dynamics of the quarks. The nonequilibrium evolution of the chiral condensate introduces a time dependent mass for the quarks. We obtained an analytic expression for the quark propagators in the mean field limit and used these to compute the triangle diagram out of equilibrium. The effective local pseudoscalar-photon vertex out of equilibrium is then extracted for quasistatic, classical electromagnetic field configurations as argued to be relevant in peripheral ultrarelativistic heavy ion collisions. An interesting aspect of the nonequilibrium calculation is that its treatment as an initial value problem has eliminated the infrared and pinch singularities found in the equilibrium situation.

From the nonequilibrium vertex we constructed the nonequilibrium equations of motion for the neutral pion in the mean field approximation, including the effective pseudoscalar-photon vertex. This equation of motion revealed potential enhancements of electromagnetic isospin breaking effects through spinodal instabilities and the nonequilibrium anomalous vertex. Thus the triangle diagram and the effective non-equilibrium pseudoscalar-photon vertex is *sensitive* to long-wavelength fluctuations of scalars and pseudoscalars through back-reaction effects.

One of the important observations (originally made in [14]) of this article is that whereas the anomaly equation and the axial Ward identity are independent of the model and the low energy sector of the theory, the pseudoscalar-photon vertex in the medium in or out of equilibrium is *non-universal*, model dependent and very sensitive to low energy physics. The next calculation to do would be a numerical study of the effective equation of motion for the neutral pion, with an eye towards answering the question addressed in [20]: can the anomalous vertex enhance the formation of neutral pion DCC regions. This would have fascinating phenomenological consequences.

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APPENDIX A: THE ANOMALOUS WARD IDENTITY OUT OF EQUILIBRIUM:

The fact that the axial Ward identity is unmodified by thermal corrections in a heat bath at finite temperature was explicitly shown in [15]. In this appendix we explicitly show via perturbation theory at 1-loop that the axial Ward identity and the anomaly equation are not modified in or out of equilibrium. This is a non-trivial statement in the real time formulation since there are, as we will see below, different contributions to the triangle diagram and anomaly equation arising from the different branches in the CTP contour of the generating functional.

We will use the point-splitting approach to derive the anomalous conservation law but implemented in real time.

The classically conserved axial current in the LSM is

$$j^a_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\tau^a\psi + \sigma\partial_\mu\pi^a - \pi^a\partial_\mu\sigma \quad , \tag{A1}$$

where the superscript (a) denotes the isospin component under consideration. The quantity of interest is the $a = 3$ component of the current which exhibits the anomalous behaviour. In the nonequilibrium CTP formulation all the operators reside on the CTP contour, carrying $+$ and $-$ labels depending on which part of the time contour they belong to and we will compute within perturbation theory the expectation value of

$$\langle \partial^\mu j^{3+}_{5\mu} \rangle . \tag{A2}$$

which is defined on the forward contour. In the subsequent steps we will omit the isospin index since it is clear which component we are focussing on. The expectation value above is of course *a priori* ill-defined because of the usual short-distance singularities in the operator product expansion. We therefore regulate this composite operator by point-splitting in a gauge invariant manner as follows,

$$\langle \partial^\mu j^{(+)}_{5\mu}(x) \rangle \equiv \lim_{\epsilon^\mu \to 0} \langle \partial^\mu \tilde{j}^{(+)}_{5\mu}(x,\epsilon) \rangle \ , \tag{A3}$$

where we have defined a new operator

$$\tilde{j}^{(+)}_{5\mu}(x,\epsilon) \equiv \tag{A4}$$

$$\bar{\psi}^+(x+\epsilon)\gamma_\mu\gamma_5\tau^3 e^{ieQ\int_x^{x+\epsilon} A(z)\cdot dz}\psi^+(x) + \sigma^+(x)\partial_\mu\pi^{0+}(x) - \pi^{0+}(x)\partial_\mu\sigma^+(x) \ .$$

The Wilson-line has been introduced to render the point-split operator gauge-invariant. This is the analogous to the treatment of the axial anomaly in spinor QED for the *vacuum* [46].

The gauge-fields can be treated simply as background fields without loss of generality. Using the equations of motion it is now easy to see that

$$\langle \partial_\mu \tilde{j}^\mu_5(x,\epsilon) \rangle = h\langle \pi^{0+}(x) \rangle + ie\langle \bar{\psi}^+(x+\epsilon)\gamma^\mu\gamma_5 Q\tau_3\psi^+(x) \rangle F_{\mu\nu}\epsilon^\nu(1+O(\epsilon)) \tag{A5}$$

The correlator on the right hand side of Eq. (A5) has short distance singularities as $\epsilon^\mu \to 0$, which we now extract via a one-loop calculation. The question that we want to answer is whether the nonequilibrium path integral introduces any new divergent contributions to the Ward identity. At one-loop we find

$$\langle \bar{\psi}^+(x+\epsilon)\gamma^\mu\gamma_5 Q\tau_3\psi^+(x)\rangle = -ie\frac{N_c}{6}\int d^4z \left\{ \mathrm{Tr}\left[S^{++}(z,x+\epsilon)\gamma^\mu\gamma_5 S^{++}(x,z)\gamma^\nu\right] + \right. \tag{A6}$$

$$\left. -\mathrm{Tr}\left[S^{-+}(z,x+\epsilon)\gamma^\mu\gamma_5 S^{+-}(x,z)\gamma^\nu\right]\right\} A_\nu(z).$$

Rewriting Eq. (A6) in terms of the spatial Fourier transforms of the Green's functions we obtain

$$-ie\frac{N_c}{6}\int dt' \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot x}\tilde{A}_\nu(\vec{q},t')\int \frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{\epsilon}} \times \tag{A7}$$

$$\left\{\mathrm{Tr}\left[S^{++}_k(t',t+\epsilon^0)\gamma^\mu\gamma_5 S^{++}_{k+q}(t,t')\gamma^\nu\right] - \mathrm{Tr}\left[S^{>}_k(t',t+\epsilon^0)\gamma^\mu\gamma_5 S^{<}(t,t')_{k+q}\gamma^\nu\right]\right\}.$$

There are *two* distinct contributions to the Ward identity out-of-equilibrium. The first one is obtained by contractions with fields on the forward time contour and involves the usual time-ordered propagators. The second term which arises from contractions with fields on the backward time contour is the new contribution that appears in the nonequilibrium case. Non-trivial modifications of the Ward identity can only arise from terms that diverge as $\epsilon \to 0$. But the latter are short distance divergences and therefore we need only to consider the UV behaviour of the integrand in Eq. (A7). As $|\vec{k}| \to \infty$, $g\sigma^2(t)$ $g\dot{\sigma}(t)$ and the temperature T_i can be ignored and the propagators in Eqs. (3.11), (3.12) and (3.14) take on their usual zero temperature short distance forms, which is simply dictated by the singularities of free-field theory in the operator product expansion of fields. Hence the UV portion of the first term can be rewritten in terms of the Fourier transformed fields as

$$ie\frac{N_c}{6}\int \frac{d^4q}{(2\pi)^4}e^{-iq.x}\tilde{A}_\nu(q)\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot\epsilon}\frac{-4i\epsilon^{\alpha\mu\beta\nu}k_\alpha q_\beta}{k^2(k+q)^2} \tag{A8}$$

which, after evaluating the divergent integral, yields

$$\frac{-ieN_c}{12\pi^2}\left(\frac{\epsilon_\alpha}{\epsilon^2}\right)\partial_\beta A_\nu(x)\epsilon^{\alpha\mu\beta\nu}. \tag{A9}$$

So the contribution of this term to the Ward identity is then

$$= \lim_{\epsilon \to 0} \left(\frac{\epsilon_\alpha \epsilon^\gamma}{\epsilon^2} \right) \frac{e^2 N_c}{24\pi^2} F_{\beta\nu} F_{\gamma\mu} \epsilon^{\alpha\mu\beta\nu} \tag{A10}$$

$$= -\frac{e^2 N_c}{96\pi^2} F_{\mu\nu} F_{\alpha\beta} \epsilon^{\mu\nu\alpha\beta} \tag{A11}$$

We now isolate the ultraviolet (the leading terms in the $\epsilon \to 0$ limit) contributions to the second term in Eq. (A7) which is due to the presence of the backward time contour

$$\frac{-ieN_c}{6}(4i\epsilon^{\alpha\mu\beta\nu}) \times \tag{A12}$$
$$\times \int \frac{d^4q}{(2\pi)^4} e^{i\vec{q}\cdot\vec{x}} \tilde{A}_\nu(\vec{q},\omega) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot\epsilon} e^{it(|\vec{k}+\vec{q}|+k)}}{4k|\vec{k}+\vec{q}|} \int dt' e^{it'(\omega - |\vec{k}+\vec{q}|)-k)} k_\alpha p_\beta$$
$$= \frac{ieN_c}{6}(4i\epsilon^{\alpha\mu\beta\nu})$$
$$\times \int \frac{d^4q}{(2\pi)^4} e^{i\vec{q}\cdot\vec{x}} \tilde{A}_\nu(\vec{q},\omega) e^{i\omega t} \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot\epsilon}}{4k|\vec{k}+\vec{q}|} 2\pi\delta(\omega - |\vec{k}+\vec{q}| - k) k_\alpha p_\beta.$$

Clearly the $\vec{k}$- integral is cut off at large momenta by the delta function and is finite in the limit $\epsilon \to 0$. Because of the presence of ϵ^ν multiplying the expression (see Eq. (A5)), this term does not survive the $\epsilon \to 0$ limit, hence there will be no further anomalous contributions to the Ward identity other than Eq. (A11). Therefore the anomalous Ward identity is in fact **unchanged** by equilibrium finite temperature or the nonequilibrium state, as expected:

$$\langle \partial_\mu j_5^{+\mu}(x) \rangle = h\langle \pi^{0+} \rangle - \frac{e^2 N_c}{96\pi^2} F_\mu F_\nu \epsilon^{\mu\nu\alpha\beta}. \tag{A13}$$

That there are no corrections in or out of equilibrium should come as no surprise since the anomalous contributions to the axial Ward identity are generated by the ultraviolet (short distance) properties of correlators and should not be affected by either temperature, finite density or the time dependence of background fields or long-wavelength fluctuations. However, since a detailed calculation in a medium in real time in or out of equilibrium was missing, this analysis provides a reassuring generalization of the vacuum result. The calculation of the expectation value for the backward branch follows the same steps and leads to the same results.

APPENDIX B: NONEQUILIBRIUM FERMION GREEN FUNCTIONS

Below we outline the construction of the nonequilibrium Green's functions for the fermions in the presence of the time-dependent background condensate $\sigma_0(t)$ [28]. We consider the solutions of the time dependent Dirac equation

$$[i\not{\partial} - g\sigma_0(t)]\psi(\vec{x},t) = 0 \tag{B1}$$

Writing the four independent solutions as

$$\mathcal{U}^{(1,2)}(\vec{x},t) = e^{i\vec{k}\cdot\vec{x}}U_k^{(1,2)}(t) \ ,$$
$$\mathcal{V}^{(1,2)}(\vec{x},t) = e^{-i\vec{k}\cdot\vec{x}}V_k^{(1,2)}(t) \ ,$$

the mode functions are found to satisfy

$$\left[i\gamma_0\frac{d}{dt} - \vec{\gamma}\cdot\vec{k} - g\sigma_0(t)\right]U_k^{(1,2)}(t) = 0 \ , \tag{B2}$$

$$\left[i\gamma_0\frac{d}{dt} + \vec{\gamma}\cdot\vec{k} - g\sigma_0(t)\right]V_k^{(1,2)}(t) = 0 \ . \tag{B3}$$

It turns out that it is convenient to write the spinors as

$$U_k^{(1,2)}(t) = \left[i\gamma_0\frac{d}{dt} - \vec{\gamma}\cdot\vec{k} + g\sigma_0(t)\right]f_k(t)u^{(1,2)} \ , \tag{B4}$$

$$V_k^{(1,2)}(t) = \left[i\gamma_0\frac{d}{dt} + \vec{\gamma}\cdot\vec{k} + g\sigma_0(t)\right]g_k(t)v^{(1,2)} \ , \tag{B5}$$

with $u^{(1,2)}$, $v^{(1,2)}$ the spinor eigenstates of γ_0 with eigenvalues $+1$, -1 respectively. The functions $f_k(t)$, $g_k(t)$ obey the second order equations

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + g^2\sigma_0^2(t) - ig\dot{\sigma}_0(t)\right]f_k(t) = 0 \ , \tag{B6}$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + g^2\sigma_0^2(t) + ig\dot{\sigma}_0(t)\right]g_k(t) = 0 \ . \tag{B7}$$

We now need to append initial conditions. We will consider the situation in which the system was in equilibrium at a temperature T_i for all times $t \leq t_i$ with the expectation value of the scalar field being $\sigma(0)$ and $\dot{\sigma}(0) = 0$. Thus the fermion mass is constant and given by $g\sigma_0(0)$. We can now impose the condition that the modes $f_k(t)$, $g_k(t)$ describe positive and negative frequency solutions for $t \leq t_i$ and, normalizing the spinor solutions to the Dirac equation to unity, we impose the following initial conditions that describe particle and antiparticle states for $t < t_i < 0$

$$f_k(t<0) = \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k(\omega_k + g\sigma_0(0))}} \; , \tag{B8}$$

$$g_k(t<0) = \frac{e^{i\omega_k t}}{\sqrt{2\omega_k(2\omega_k + g\sigma_0(0))}} \; , \tag{B9}$$

$$\omega_k = \sqrt{\vec{k}^2 + g^2\sigma_0^2(0)} \; . \tag{B10}$$

Equations (B7) with these boundary conditions imply that

$$g_k(t) = f_k^*(t) \; . \tag{B11}$$

1. Initial temperature $T_i = 0$:

The necessary ingredients for the zero temperature fermionic Green's functions are the following

$$S^{>}_{0k}(t,t') = -i \sum_{\alpha=1,2} U^{\alpha}_{\vec{k}}(t)\bar{U}^{\alpha}_{\vec{k}}(t') \; , \tag{B12}$$

$$S^{<}_{0k}(t,t') = i \sum_{\alpha=1,2} V^{\alpha}_{-\vec{k}}(t)\bar{V}^{\alpha}_{-\vec{k}}(t') \; . \tag{B13}$$

In the standard Dirac representation for the γ matrices, we find

$$S^{>}_{0k}(t,t') = \tag{B14}$$
$$-if_k(t)f_k^*(t')\left[\mathcal{W}_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} + g\sigma_0(t)\right]\left(\frac{1+\gamma_0}{2}\right)\left[\mathcal{W}^*_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} + g\sigma_0(t')\right] \; ,$$

$$S^{<}_{0k}(t,t') = \tag{B15}$$
$$-if_k^*(t)f_k(t')\left[\mathcal{W}^*_k(t)\gamma_0 - \vec{\gamma}\cdot\vec{k} - g\sigma_0(t)\right]\left(\frac{1-\gamma_0}{2}\right)\left[\mathcal{W}_k(t')\gamma_0 - \vec{\gamma}\cdot\vec{k} - g\sigma_0(t')\right] \; ,$$

$$\mathcal{W}_k(t) = i\frac{\dot{f}_k(t)}{f_k(t)} \; . \tag{B16}$$

With these basic results in place, the nonequilibrium fermionic Green's functions can be constructed using Eqs. (3.7), (3.7), (3.9) and (3.10). The fact that the equality $\mathrm{Tr}\, S^{++}_{0k}(t,t) = \mathrm{Tr}\, S^{--}_{0k}(t,t) = \mathrm{Tr}\, S^{>}_{0k}(t,t) = \mathrm{Tr}\, S^{<}_{0k}(t,t)$ is satisfied (where the trace is over Dirac indices) provides an important check. The mode functions $f_k(t)$ satisfy yet another important property which is a consequence of probability conservation and can also be checked explicitly [28],

$$\left[-i\dot{f}^*_k(t) + g\sigma_0(t)f^*_k(t)\right]\left[i\dot{f}_k(t) + g\sigma_0(t)f_k(t)\right] + k^2\, f^*_k(t)f_k(t) = 1 \; . \tag{B17}$$

2. Initial temperature $T_i \neq 0$:

When the initial temperature is not zero, the CTP contour extends into the complex plane with a leg that runs from t_0 to $t_0 - i\beta$.. The fermionic fields and Green functions then satisfy antiperiodic boundary conditions:

$$S^{>}_{k}(t_0 - i\beta, t) = -S^{<}_{k}(t_0, t). \tag{B18}$$

Incorporating these boundary conditions then leads to the following formulae for the nonequilibrium fermion propagators at finite initial temperature:

$$S^{>}_{\vec{k}}(t,t') = S^{>}_{0\vec{k}}(t,t')\left[1 - n_f(\omega_k)\right] - S^{<}_{0\vec{k}}(t,t')n_f(\omega_k) \ , \tag{B19}$$

$$S^{<}_{\vec{k}}(t,t') = S^{<}_{0\vec{k}}(t,t')\left[1 - n_f(\omega_k)\right] - S^{>}_{0\vec{k}}(t,t')n_f(\omega_k) \ , \tag{B20}$$

$$\omega_k = \sqrt{\vec{k}^2 + g^2\sigma^2(0)} \ , \qquad\qquad n_f(\omega_k) = \frac{1}{e^{\beta\omega_k} + 1} \ .$$

In equilibrium, i.e. when σ_0 is a constant function of time, the fermion propagators are:

$$\begin{aligned} S^{>}_{\vec{k}}(t,t') &= -\frac{i}{2\omega_k}\left[e^{-i\omega_k(t-t')}(\not{k} + m_\psi)(1 - n_f(\omega_k)) + e^{i\omega_k(t-t')}\gamma_0(\not{k} - m_\psi)\gamma_0 n_f(\omega_k)\right] \ , \\ S^{<}_{\vec{k}}(t,t') &= \frac{i}{2\omega_k}\left[e^{-i\omega_k(t-t')}(\not{k} + m_\psi)n_f(\omega_k) + e^{i\omega_k(t-t')}\gamma_0(\not{k} - m_\psi)\gamma_0(1 - n_f(\omega_k))\right] \ . \end{aligned} \tag{B21}$$

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